

Groups with undecidable word problem and almost quadratic Dehn function

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with an Appendix by M.V.Sapir¹

Abstract

We construct a finitely presented group with undecidable word problem and with Dehn function bounded by a quadratic function on an infinite set of positive integers.

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1 Introduction

1.1 Formulation of results

The minimal non-decreasing function $f(n): \mathbb{N} \rightarrow \mathbb{N}$ such that every word w vanishing in a group $G = \langle A \mid R \rangle$ and having length $\|w\| \leq n$, freely equal to a product of at most $f(n)$ conjugates of relators from R is called the *Dehn function* of the presentation $G = \langle A \mid R \rangle$ [5]. By van Kampen's Lemma, $f(n)$ is equal to the maximal area of minimal diagrams Δ with perimeter $\leq n$. (See Subsection 5.1 for the definitions.) For *finitely presented* groups (i.e., both sets A and R are finite) Dehn functions are usually taken up to equivalence to get rid of the dependence on a finite presentation for G (see [9]). To introduce this *equivalence* \sim , we write $f \preceq g$ if there is a positive integer c such that $f(n) \leq cg(cn) + cn$ for any $n \in \mathbb{N}$. Two non-decreasing functions f and g on \mathbb{N} are called equivalent if $f \preceq g$ and $g \preceq f$.

A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is called *almost quadratic* if there exists a constant $C > 0$ and an infinite set of integers B , such that $f(b) < Cb^2$ for all $b \in B$.

It is well known that a finitely presented group has undecidable word problem if and only if its Dehn function $d(n)$ is not bounded by a recursive function (and if and

only if $d(n)$ is not recursive itself; see [4], [3]), whence for every recursive function $f(n)$, $d(n) > f(n)$ for infinitely many values of n . The main result of this paper shows that a non-recursive Dehn function can be almost quadratic at the same time.

Theorem 1.1. *There exists a finitely presented group G with undecidable word problem and almost quadratic Dehn function.*

Note that “almost quadratic” is the smallest Dehn function one can get, because if the Dehn function of a finitely presented group is $o(n^2)$ on some infinite set of integers, then the group is hyperbolic and its Dehn function is linear (this follows from Gromov [5, 6.8.M] or Bowditch [2]).

By Theorem 1.1, for some infinite set B of natural numbers b , the Dehn function of G satisfies the condition $f(b) < Cb^2$ for some constant C . Notice that the set B is not recursive or even recursively enumerable although its complement is recursively enumerable. Indeed, if $f(b) \geq Cb^2$, then there exists a word w of length $\leq b$ which is equal to 1 in the group, but which is not the boundary label of any van Kampen diagram with less than Cb^2 cells; all diagrams with this number of cells and boundary length at most b can be enumerated; and all words that are equal to 1 in the group can be enumerated too. Moreover, B cannot contain any infinite recursively enumerable subset (i.e. it is *immune* in the terminology of [8]). Indeed if B contains an infinite recursively enumerable set enumerated by a Turing machine M , then in order to check if a word w is 1 in G (and solve the word problem in G) we would do the following: wait till M produces a word w' longer than w . Then the area of the minimal van Kampen diagram for w cannot exceed $C||w'||^2$ (here and below $||w||$ denotes the length of the word w), and it would remain to check all diagrams of that area. Thus although B exists and is infinite, there is no algorithm to find any infinite part of it.

As a corollary of Theorem 1.1 and the results of [16] and [6] we get

Corollary 1.2. *The group G from Theorem 1.1 has a simply connected and a non-simply connected asymptotic cone.*

Indeed the asymptotic cone corresponding to the sequence B discussed in the previous paragraph is simply connected by [16]. On the other hand all asymptotic cones of G cannot be simply connected because that would imply decidability of the word problem in G by [6].

Undecidability of conjugacy problem is easier to achieve than undecidability of the word problem.

Theorem 1.3. *There exists a finitely presented (multiple) HNN extension M of a free group with finitely generated associated subgroups and with Dehn function $f(n)$ such that:*

1. *The conjugacy problem is undecidable in M ;*
2. *There is an infinite set $N_1 \subseteq \mathbb{N}$, such that for some constant C we have $f(n) < Cn^2$ for every $n \in N_1$;*
3. *For every n , $f(n) \leq Cn^3$.*

Remark 1.4. Probably the first example of an almost quadratic but not quadratic Dehn function of a finitely presented group was constructed in [12]. However that function is $O(n^2 \log n / \log \log n)$ which is not much bigger than a quadratic function. A slight modification of the proofs of the present paper provides us with a *recursive* almost quadratic

Dehn function $d(n)$ rapidly increasing on some infinite subset N_2 of \mathbb{N} . (For example, almost quadratic $d(n)$ is at least exponential on N_2 and at most exponential on the entire \mathbb{N} ; see Theorem 13.5 and Remark 13.6 for details.) The difference is that the proof of Theorem 1.1 uses Sapir's Theorem 14.1, but the recursive examples are independent of it.

Remark 1.5. Using [19] and Theorem 14.1 one can easily obtain a weaker version of Theorem 1.1 replacing “almost quadratic” by “almost polynomial”. Recall that a function $f: \mathbb{N} \rightarrow \mathbb{N}$ is superadditive if $f(m+n) \geq f(m) + f(n)$ for any $m, n \in \mathbb{N}$. The superadditive closure $\bar{f}(n)$ of a function $f(n)$ is given by the formula $\bar{f}(n) = \max(f(n_1) + \dots + f(n_k))$ over all non-negative partitions $n = n_1 + \dots + n_k$. If a Turing machine M accepts a language L with at least linear time function $T(n)$, and $T(n)^4$ is equivalent to a superadditive function, then by Theorem 1.3 [19], there is a finitely presented group $G(M)$ with Dehn function $d(n)$ equivalent to $T(n)^4$. But in fact, it is proved in [19] that omitting the assumption that $T(n)^4$ is superadditive, we have inequalities $T(n)^4 \preceq d(n) \preceq \overline{T(n)^4}$. Thus it suffices to construct a Turing machine M with non-recursive but “almost linear” time function $T(n)$. The existence of such a machine follows from Theorem 14.1. (Moreover, one can derive from Theorem 14.1 that $\overline{T(n)^4}$ is “almost n^4 ”.)

Reducing to “almost quadratic” (as in Theorem 1.1) requires a new approach. The S -machine we are going to use will be different from [19], and the analysis of diagrams will be much more delicate. The main reason for the difficulties arising here is that the “almost quadratic” property is unimprovable. For example, the *cubic* upper bound of the Dehn function of the group M is obvious in [19] (see also Step 1 in the proof of Lemma 13.1 below), but the main contents of our paper focus on M , starting with the properties of the machine defining M and ending with new quadratic invariants of the diagrams called mixture(s) on their boundaries. In the next subsection of the introduction, we discuss the outline of the proof of Theorem 1.1, and some ideas needed in its proof.

1.2 A short description of the proof of Theorem 1.1

Relations of a finitely presented group with undecidable word problem simulate the commands of a Turing machine M_0 with undecidable halting problem, and as in the works of P. Novikov, W. Boone and many other authors (see [17], [18]), one has to properly code the work of a Turing machine in terms of group relations. To obtain an almost quadratic Dehn function of a group G , we must start with a machine having almost linear time function (but which is not bounded from above by any recursive function). Thus we can just demand that the lengths of words accepted by M_0 form a very sparse subset of positive integers $B \subset \mathbb{N}$. As a measure of how sparse B is, M.V. Sapir gives the following exact definition.

Let X be a recursively enumerable (r.e.) language in the binary alphabet recognized by a Turing machine M . If $w \in X$ then the *time* of w (denoted $\text{time}(w)$ or $\text{time}_M(w)$) is, by definition, the minimal time of an accepting computation of M with input w . For an increasing function $h: \mathbb{N} \rightarrow \mathbb{N}$, a number $m \in \mathbb{N}$ is called *h -good* for M if for every word $w \in X$ of length $< m$, we have $h(\text{time}(w)) < m$.

For our estimates, it suffices to start with a Turing machine M_0 recognizing a r.e. non-recursive set X such that the set of all f -good numbers for M_0 is infinite, where f is arbitrary double exponential function. Such a machine is constructed in the Appendix written by M.V. Sapir (see Theorem 14.1).

Group relations always interpret the symmetrization of a machine. Thus as a preliminary step, one has to add the inverse commands, in spite of the fact that the machine M_0 is replaced by a non-deterministic machine M_1 . (Of course, one should be concerned that the symmetrization preserves some basic characteristics of the machine.) However the interpretation problem for groups remains much harder than for semigroups even after modifying the machine because the group theoretic simulation can execute unforeseen computations with non-positive words. Boone and Novikov secured the positiveness of admissible configurations with the help of an additional ‘quadratic letter’ (see [17], Ch.12). However this old trick implies that the constructed group G contains Baumslag - Solitar groups $B_{1,2}$ and has at least exponential Dehn function. Since we want to obtain almost quadratic Dehn function, we use a new approach suggested in [19]. Invented by Sapir, S-machines can work with non-positive words on the tapes. Here we use an a composition M_2 of the symmetric machine M_1 with an ‘adding machine’ $Z(A)$ introduced in [15]. This S-machine is equivalent to M_1 .

Here we have to guarantee at least two important properties of the machine M_2 (since the violation of either of them makes the Dehn function of the group M_2 non-almost quadratic): a reduced computation of M_2 does not repeat the same configuration twice, and every *accepting* computation of M_2 is uniquely determined by the initial configuration (though M_2 is highly non-deterministic).

Every S -machine is, on the one hand, a rewriting system and, on the other hand, it can be treated as a multiple HNN-extension of a free group (see [15] or [18]). But when one takes an S -machine as an HNN-extension, then the number of working heads can be arbitrarily large and their order on the common tape can be non-standard. Therefore, as in [19] or [15], we have to extend the set of admissible words for the machine treated as a rewriting system. This makes the control of arbitrary computation difficult. (There was no need for such accurate control in [19] or [15].) Hence we are forced to introduce auxiliary control heads which are called upon to examine the order of heads after and/or before the application of every rule of the machine M_2 . The obtained machine M_3 is better than M_2 because it is able to accomplish only ‘simple’ computations with non-standard disposition of the heads.

When we introduce a new machine, then clearly, we should check that it inherits the important properties of the machines studied earlier. In particular, M_3 inherits the language accepted by M_1 . The next modification is the machine M_4 which has two additional tapes with histories of what M_3 computes. For any computation with the standard order of heads of M_4 (the notion of standard base of M_4 is given in subsection 4.3), we prove that either the time T of this computation is ‘close’ to the time T_i of some computation accepting a word from the sparse set provided by Theorem 14.1, or the space on the ‘historical’ tapes at the beginning or at the end of the computation is bounded from below by a linear function of T . In the latter case we have a quadratic upper estimate for the area of the trapezium corresponding to the computation. Here the definition of trapezium as a special van Kampen diagram is borrowed from [15], and Lemma 5.10 translates the machine language to the diagram language.

Finally, the machine M is a union of many copies and mirror copies of M_4 working in a parallel way. The corresponding HNN-extension M is the group from Theorem 1.3. The accept word of the machine M is called the hub. There is no algorithm deciding if a given word in the generators of M is conjugate to the hub. The usual adding of the hub relation to the list of defining relations of M (as in [17], and many papers) provides us with the group G for Theorem 1.1. As in [19] or [11] the hub relation has many copies

of the accept words of M_4 . This makes the hub graph (with vertices in hub cells; see Subsection 5.3) associated with a van Kampen diagram, hyperbolic, and this is used in Lemmas 5.18 and 5.19. The mirror symmetry of the hub is used for the surgery removing a hub (see Subsection 12.2).

Unsolvability of the halting problem for the S-machine M immediately implies that the Dehn function $d(n)$ of the group G is not bounded from above by any recursive function. Other precautions used in the construction of the machines M_0, \dots, M_4, M eliminate a number of visual obstacles standing in the way of the almost quadratic property for the Dehn functions of M and G . (For instance, if one uses only one historic tape for M_4 or the arrangement of the historic heads is different, then the almost quadratic estimate is not achieved by our model.) But how can one *prove* this property ?

The areas of diagrams whose perimeters are close to some numbers T_i mentioned above can be non-recursively high in comparison with their perimeters. Therefore one has to consider a diagram Δ whose perimeter n is far from the infinite increasing sequence $\{T_1, T_2, \dots\}$, for example, $\exp T_{i-1} < n < T_i$ for some i . Unlike a trapezium, an arbitrary diagram has irregular structure. Therefore we want to find some more regular pieces to cut them off and then use an induction on the perimeter n .

Indeed assume that there is a simple path \mathbf{y} in Δ cutting up Δ into two subdiagrams Δ_1 and Δ_2 with boundary paths \mathbf{yz} and $\mathbf{y}^{-1}\mathbf{z}'$, resp., where \mathbf{zz}' is the boundary of Δ . Assume that $|\mathbf{z}| > |\mathbf{y}|$ (where $|\mathbf{x}|$ is the length of a path \mathbf{x}) and moreover, $\text{Area}(\Delta_1) \leq C|\mathbf{y}|(|\mathbf{z}| - |\mathbf{y}|)$, where the positive constant C does not depend on the diagrams. Then it is easy to see that the quadratic estimate $\text{Area}(\Delta_2) \leq C|\mathbf{yz}|^2$ for the subdiagram Δ_2 with perimeter $< n$ together with the estimate for $\text{Area}(\Delta_1)$ give

$$\text{Area}(\Delta) \leq \text{Area}(\Delta_1) + \text{Area}(\Delta_2) \leq C|\mathbf{zz}'|^2 = Cn^2$$

as required. Thus we are looking for pieces whose area can be estimated as for Δ_1 .

First of all, among such ‘good’ pieces, we have so called rim θ -bands with a restriction on the length (but here we have to change the usual combinatorial metric by the metric, where the generators from the tape alphabet of the machine M are much shorter than other generators of the groups M and G). The ‘good’ pieces of second type (again, under some restrictions) are combs defined in Section 7 (and introduced earlier in [15]).

The upper bound of the form $C|\mathbf{y}|(|\mathbf{z}| - |\mathbf{y}|)$ works for many types of combs but unfortunately, it is false for other combs whose areas must also be estimated. We have found another quadratic invariant of the boundaries of the diagrams, called mixture. In Section 6, we associate a two-colored necklace with the boundary \mathbf{p} of Δ . The black and white beads of this necklace correspond to different types of edges in \mathbf{p} . To obtain the mixture $\mu(\Delta)$ one calculates the number of pairs of white beads separated in \mathbf{p} by black ones. (Another quadratic invariant, called dispersion, was introduced and applied earlier in [15], but the dispersion depends on the whole diagram and works for hub free diagrams while the mixture depends on the boundary label only and works for arbitrary diagram over G .)

The important observation is that for many types of subcombs Δ_1 , we have inequalities $\text{Area}(\Delta_1) \leq C|\mathbf{y}|(|\mathbf{z}| - |\mathbf{y}|) + \mu(\Delta_1)$, and $\mu(\Delta_2) \leq \mu(\Delta) - \mu(\Delta_1)$. This was a breakthrough which inspired the confidence that the whole project would be completed. However the original mixture cannot help in case of some special combs. Therefore we have to consider boundary necklaces of 3 different types. The different mixtures help to estimate the areas of different combs. But one of these mixtures helps in some cases and

can be negative in some other cases, which causes a problem for our induction. Therefore we use a weighted linear combination of 3 mixtures in the Lemma 11.8 summarizing our estimates of comb areas. Hence we have to estimate the behavior of these mixtures in different situations, which makes a number of comb lemmas complicated, and the comb part of the paper is the hardest one.

Then we consider a diagram Δ with hubs. Due to hyperbolicity of the hub structure mentioned above, there is a hub Π such that almost all ‘spokes’ starting on Π end on the boundary $\partial\Delta$, and they bound (together with $\partial\Delta$ and $\partial\Pi$) a subdiagram Ψ without hubs. Now we are able to remove redundant combs and rim bands from Ψ . The remaining *crescent* $\tilde{\Psi}$ together with Π can be cut off by a relatively short cutting path. (Thus, one can also induct on the number of hubs in Δ .) As in [19], our surgery uses the mirror symmetry of the hub relation, but our inequalities are more delicate here than those used for the ‘snowball decomposition’ in [19] since we aim for almost quadratic bounds. Again we estimate the area of the removed part in terms of the reduction of the perimeter, of the mixtures, and more. To complete the proof, we take into account that the auxiliary parameters are quadratically bounded with respect to the perimeter of a diagram.

The author is aware that such a long proof can be arduous to the reader. Making our apology we collect all the definitions and terms at the end of the paper (see Subject Index) and insert many pictures and brief comments throughout the text. Besides, Lemma 5.17 reformulates all machine properties we need in terms of van Kampen diagrams so that the machine constructions can be forgotten after one has read that lemma.

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2 A Turing machine

2.1 Definitions and notations related to Turing machines

In this section, we collect all information about Turing machines that we need in the proof of our main results.

As usual we consider words as sequences of symbols from some alphabet X .

We shall use the following standard notation for Turing machines. A (multi-tape) *Turing machine* has k tapes and k heads observing the tapes. One can view it as a structure

$$M = \langle I, Y, Q, \Omega, \Theta \rangle,$$

where I is the input alphabet, Y is the tape alphabet ($I \subseteq Y$), $Q = \sqcup Q_i, i = 1, \dots, k$ is the set of states of the heads of the machine (and \sqcup denotes disjoint union),

$\Omega = \{\alpha_1, \omega_1, \dots, \alpha_k, \omega_k\}$ is the set of left and right markers of the tapes, and Θ is a set of commands.

The leftmost (the rightmost) square on the i -th tape is always marked by α_i (by ω_i). The head is placed between two consecutive squares on the tape. A *configuration* of the i -th tape of a Turing machine is a word $\alpha_i u q v \omega_i$, where $q \in Q_i$ is the current state of the head of that tape, u is the word in Y to the left of the head and v is the word in Y to the right of the head, and so the word written on the entire tape is uv ; so we do not include α_i, ω_i and the state letter when we talk about the word written on the tape.

At every moment the head of each tape observes two letters on that tape: the last letter of u (or α_i) and the first letter of v (or ω_i).

A *configuration* U of a Turing machine is the word

$$U_1 U_2 \dots U_k,$$

where U_i is the configuration of tape i . We shall omit the indices i of α_i and ω_i for the sake of brevity.

Assuming that the Turing machine is recognizing, we can define input configurations and accepted (stop) configurations. An *input configuration* is a configuration where the word written on the first tape is in I , all other tapes are empty, the head on the first tape observes the right marker ω , and the states of all tapes form a special *start* k -vector \vec{s}_1 . An *accept (or stop) configuration* is any configuration where the state vector for a special k -vector \vec{s}_0 , the *accept vector* of the machine. We shall always assume (as can be easily achieved) that in the accept configuration of a Turing machine every tape is empty.

A transition (*command*) of a Turing machine is given by the states of the heads and some of the $2k$ letters observed by the heads. As a result of a transition we replace some of these $2k$ letters by other letters, insert new squares in some of the tapes and may move the heads one square to the left (right) with respect to the corresponding tapes.

For example in a one-tape machine, every transition is of the following form:

$$uqv \rightarrow u'q'v',$$

where u, v, u', v' are letters (could be end markers) or empty words. The only constraint is that the result of applying the substitution $uqv \rightarrow u'q'v'$ to a configuration word must be a configuration word again, in particular the end markers cannot be deleted or inserted. This command means that if the state of the head is q , u is written to the left of q and v is written to the right of q , then the machine must replace u by u' , q by q' and v by v' .

For a general k -tape machine, a command is a vector

$$[U_1 \rightarrow V_1, \dots, U_k \rightarrow V_k],$$

where $U_i \rightarrow V_i$ is a command of a 1-tape machine, the elementary commands (also called *parts of the command*) $U_i \rightarrow V_i$ are listed in the order of tape numbers. In order to execute this command, the machine checks if U_i is a subword of the configuration of tape i ($i = 1, \dots, k$), and then replaces U_i by V_i .

Notice that for every command $[U_1 \rightarrow V_1, \dots, U_k \rightarrow V_k]$, the vector $[V_1 \rightarrow U_1, \dots, V_k \rightarrow U_k]$ is also a command of a Turing machine. These two commands are called *mutually inverse*. A Turing machine is called *symmetric* if for every command of that machine, the inverse is also a command of the machine. If a Turing machine is symmetric, we shall always consider a division of the set of its commands Θ into two disjoint subsets, positive and negative commands: $\Theta = \Theta^+ \sqcup \Theta^-$, so that the inverses of commands in Θ^+ are in Θ^- and vice versa.

We will assume that only input configurations of a Turing machine involve the state letters from \vec{s}_1 and only one (positive) command θ_{start} is applicable to the input configurations. Similarly, we assume that there is a unique accept configuration \vec{s}_0 and a unique (positive) accepting command θ_{accept} .

A *computation* is a sequence of configurations $C_0 \rightarrow \dots \rightarrow C_n$ such that for every $0 = 1, \dots, n - 1$ the machine passes from C_i to C_{i+1} by applying one of the commands

from Θ . A configuration C is said to be *accepted* by a machine M if there exists at least one computation which starts with C and ends with the accept configuration.

A word u over I is said to be *accepted* by the machine if the corresponding input configuration is accepted. The set of all accepted words over the input alphabet I is called the *language accepted (recognized) by the machine*.

Let $C = C_0 \rightarrow \dots \rightarrow C_n$ be a computation of a machine M such that for every $j = 0, \dots, n-1$ the configuration C_{j+1} is obtained from C_j by a command θ_{j+1} from Θ . Then we call the word $\theta_1 \dots \theta_n$ the *history* of this computation. The number n will be called the *time* (or *length*) of the computation.

Remark 2.1. Note that we can (and will) assume that in every command $[u_1 q_1 v_1 \rightarrow u'_1 q'_1 v'_1, \dots, u_k q_k v_k \rightarrow u'_k q'_k v'_k]$ the sum of numbers of letters from the tape alphabet Y in all u_i, u'_i, v_i, v'_i , $i = 1, \dots, k$, is at most 1. Indeed, this can be achieved by subdividing a command in the standard way. For example, a command $[aq \rightarrow bq']$ is replaced by two commands $[aq \rightarrow q''], [q'' \rightarrow bq']$, where q'' is a new state letter.

It is convenient to consider *empty computations* consisting of one word W . The history of an empty computation is the empty word, the start and end words of this computation are equal to W . We do not only consider *deterministic* Turing machines, for example, we allow several transitions with the same left side. For example, most symmetric Turing machines are not deterministic.

2.2 A conversion of a deterministic Turing machine into a symmetric Turing machine

At first, let us add some useful properties to a machine.

Lemma 2.2. *Let M_0 be a deterministic Turing machine recognizing a set of words X . Then there exists a deterministic Turing machine M_1 which recognizes X and such that*

- (a) *If $W \equiv W'$ (i.e. these two words are letter-for-letter equal) for a computation $W \rightarrow \dots \rightarrow W'$, then this computation is empty.*
- (b) *The property from Remark 2.1 ("at most one tape letter") holds for every command of M_1 .*
- (c) *The state letters from the start vector \vec{s}_1 (from the accept vector \vec{s}_0) occur in the left-hand side of a unique command $U \rightarrow V$ of M_1 and do not occur in the right-hand side of any command (resp., occur in the right-hand side of a unique command $U' \rightarrow V'$ and do not occur in the left-hand side of any command)*
- (d) *The letters used on different tapes are from disjoint alphabets. The letters to the left and to the right of the head of any tape are from disjoint alphabets.*

Proof. Let W be a configuration of M_0 . The general form of a configuration of M_1 will be $W\alpha\tau^l q_{k+1}\omega$, that is the machine M_1 has one more tapes than M_0 . The last tape contains a (non-negative) power of a special tape letter τ . The set of state letters is then increased by one component $\{q_{k+1}\}$. At the beginning the last tape is empty. The machine will execute M_0 on its tapes, adding new τ on the last tape after every step of computation. After M_0 accepts, M_1 erases the last tape and stops. The last tape guarantees Property

(a)). In order to get Property (b) of M_1 , we apply the trick from Remark 2.1 since it does not violate Property (a).

Property (c) of M_1 for the start command follows from the same property of M_0 , and we can define the accept command of M_1 so that Property (c) also holds for it.

In order to obtain Property (d), we use different copies of the tape alphabet for different tapes, and moreover, we use different copies from the left and from the right of each head. \square

If M_1 is a deterministic Turing machine, satisfying the properties of Lemma 2.2, with the set of commands Θ such that $\Theta \cap \Theta^{-1} = \emptyset$, then let $\text{Sym}(M_1)$ be the Turing machine with the set of commands $\Theta \sqcup \Theta^{-1}$ and the same sets of state and tape letters. The division of the commands of $\text{Sym}(M_1)$ into positive and negative is natural: the commands of M_1 are positive, their inverses are negative. The computation of $\text{Sym}(M_1)$ is called *reduced* if its history is a reduced word. Clearly, every computation can be made reduced (without changing the start or end configurations of the computation) by removing consecutive mutually inverse commands.

Lemma 2.3. *The Turing machine $\text{Sym}(M_1)$ satisfies the following properties.*

- (a) *Every command of $\text{Sym}(M_1)$ satisfies Property (b) of Lemma 2.2.*
- (b) *Every reduced history of computation of $\text{Sym}(M_1)$ has the form $H_1 H_2^{-1}$, where H_1, H_2 consist of positive commands.*
- (c) *$\text{Sym}(M_1)$ satisfies Properties (a), (c) (for positive commands), and (d) of Lemma 2.2*
- (d) *The language recognized by $\text{Sym}(M_1)$ is X .*
- (e) *For every $W \in X$ there exists only one accepting computation of $\text{Sym}(M_1)$. It is equal to the computation accepting W by M_1 , and if M_1 is given by Lemma 2.2, then the length of this computation is big- O of the length of the accepting computation of M_0 with input W .*

Proof. Property (a) is obvious. Property (b) follows immediately from the fact that in a reduced computation, a command from Θ^{-1} cannot be followed by a command from Θ (since M_1 is deterministic). Properties (c), (d), (e) follow from (b). \square

3 S -machines

3.1 S -machines as rewriting systems

There are several interpretations of S -machines in groups, the most complicated so far is in [14]. Another interpretation is given in [15], and in fact it is probably the easiest way to view an S -machine as a group that is a multiple HNN extension of a free group. Here we use a definition which is close to the original one [19] and define S -machines as rewriting systems, similar to Turing machines. S -machines work with words in group alphabets and they are almost "blind", i.e., the heads do not observe the tape letters. But the heads can "see" each other if there are no tape letters between them. We will use the following precise definition of an S -machine \mathcal{S} .

A hardware of an S -machine \mathcal{S} is a pair (Y, Q) , where $Q = \sqcup_{i=0}^N Q_i$ and $Y = \sqcup_{i=1}^N Y_i$ (for convenience we always set $Y_0 = Y_{N+1} = \emptyset$). The elements from Q are called *state letters*, the elements from Y are *tape letters*. The sets Q_i (resp. Y_i) are called *parts* of Q (resp. Y).

The *language of admissible words* consists of reduced words of the form

$$W \equiv q_1^{\pm 1} u_1 q_2^{\pm 1} \dots u_k q_{k+1}^{\pm 1}, \quad (3.1)$$

where every subword $q_i^{\pm 1} u_i q_{i+1}^{\pm 1}$ either

- (1) belongs to $(Q_j F(Y_{j+1}) Q_{j+1})^{\pm 1}$ for some j and $u_i \in F(Y_{j+1})$, where $F(Y_i)$ is the set of reduced group words in the alphabet $Y_i^{\pm 1}$, or
- (2) has the form quq^{-1} for some $q \in Q_j$ and $u \in F(Y_{j+1})$, or
- (3) is of the form $q^{-1}uq$ for $q \in Q_j$ and $u \in F(Y_j)$.

We shall follow the tradition of calling state letters q -letters and tape letters a -letters, even though we shall use other letters as state or tape letters. Usually parts of the set Q of state letters are denoted by capital letters. For example, a set K would consist of letters k with various indices. Then we shall say that letters in K are k -letters or K -letters.

If a group word W over $Q \cup Y$ has the form $q_1 u_1 q_2 u_2 \dots q_s$, and $q_i \in Q_{j(i)}^{\pm 1}$, $i = 1, \dots, s$, u_i are group words in Y , then we shall say that the *base* of W is $\text{base}(W) \equiv Q_{j(1)}^{\pm 1} Q_{j(2)}^{\pm 1} \dots Q_{j(s)}^{\pm 1}$. Here Q_i are just letters, denoting the parts of the set of state letters. Note that the base is not necessarily a reduced word, and the last equality is in the free semigroup. The subword of W between the $Q_{j(i)}^{\pm 1}$ -letter and the $Q_{j(i+1)}^{\pm 1}$ -letter will be called a $Q_{j(i)}^{\pm 1} Q_{j(i+1)}^{\pm 1}$ -sector of W . A word can certainly have many $Q_{j(i)}^{\pm 1} Q_{j(i+1)}^{\pm 1}$ -sectors.

For aesthetic reasons, we shall substitute the capital names of parts of Q by the corresponding small letters. For example, if $t \in T, k \in K, \dots$, we shall say that the base is $tk\dots$, that is the state letters in W start with a t -letter, followed by a k -letter, and so on. Usually instead of specifying the names of the parts of Q and their order as in $Q = Q_0 \sqcup Q_2 \sqcup \dots \sqcup Q_N$, we say that *the standard base* of the S -machine is $Q_0 \dots Q_N$ or $q_0 \dots q_N$.

The S -machine also has a set of *rules* Θ . Every $\theta \in \Theta$ is assigned two sequences of reduced words: $[U_0, \dots, U_N]$, $[V_0, \dots, V_N]$, and a subset $Y(\theta) = \cup Y_i(\theta)$ of Y , where $Y_i(\theta) \subseteq Y_i$.

The words U_i, V_i satisfy the following restriction:

(**) For every $i = 0, \dots, N$, the words U_i and V_i have the form

$$U_i \equiv v_i q_i u_{i+1}, \quad V_i \equiv v'_i q'_i u'_{i+1},$$

where q_i, q'_i are state letters in Q_i , u_{i+1} and u'_{i+1} are words in the alphabet $Y_{i+1}(\theta)^{\pm 1}$, v_i and v'_i are words in the alphabet $Y_i(\theta)^{\pm 1}$.

The pair of words U_i, V_i is called a *part* of the rule, and is denoted $U_i \rightarrow V_i$.

We will denote the rule θ by $[U_1 \rightarrow V_1, \dots, U_N \rightarrow V_N]$. This notation contains all the necessary information about the rule except for the sets $Y_i(\theta)$. In most cases it will be clear what these sets are. In the S -machines used in this paper, the sets $Y_i(\theta)$ will be equal to either Y_i or \emptyset . By default $Y_i(\theta) = Y_i$. If $Y_{i+1}(\theta) = \emptyset$, then the corresponding component $U_i \rightarrow V_i$ will be denoted $U_i \xrightarrow{\ell} V_i$ and we shall say that the rule *locks* the

$Q_i Q_{i+1}$ -sectors. In that case we always assume that U_i, V_i do not have tape letters to the right of the state letters, i.e., it has the form $v_i q_i \xrightarrow{\ell} v'_i q'_i$. Similarly, these words have no tape letters to the left of the state letters if the $Q_{i-1} Q_i$ -sector is locked by the rule.

Every S -rule $\theta = [U_1 \rightarrow V_1, \dots, U_s \rightarrow V_s]$ has an inverse $\theta^{-1} = [V_1 \rightarrow U_1, \dots, V_s \rightarrow U_s]$ which is also a rule of \mathcal{S} ; we set $Y_i(\theta^{-1}) = Y_i(\theta)$. We always divide the set of rules Θ of an S -machine into two disjoint parts, Θ^+ and Θ^- such that for every $\theta \in \Theta^+$, $\theta^{-1} \in \Theta^-$ and for every $\theta \in \Theta^-$, $\theta^{-1} \in \Theta^+$ (in particular $\Theta^{-1} = \Theta$, that is any S -machine is symmetric). The rules from Θ^+ (resp. Θ^-) are called *positive* (resp. *negative*).

To apply an S -rule θ to an admissible word (3.1) W means to check if all tape letters of W belong to the alphabet $Y(\theta)$ and then, if W satisfies this condition, to replace simultaneously subwords $U_i^{\pm 1}$ by subwords $V_i^{\pm 1}$ ($i = 1, \dots, k+1$) and to trim a few first and last a -letters (to obtain an admissible word starting and ending with q -letters). This replacement is allowed to perform in the form $q_i^{\pm 1} \rightarrow (v'_{i-1} v_i^{-1} q'_i u_i^{-1} u'_i)^{\pm 1}$ followed by the reducing of the resulted word. The following convention is important in the definition of S -machine: *After every application of a rewriting rule, the word is automatically reduced. The reducing is not considered a separate step of an S -machine.*

If a rule θ is applicable to an admissible word W (i.e., W belongs to the domain of θ) then the word W is called θ -admissible. The definitions of computation, its history, input admissible words, are similar to those for Turing machines. Similarly, we sometimes choose a distinguished *stop word* W_0 from the free group $F(Q)$. It will always have the standard base (and no a -letters). We say that a word $W \in F(Q \cup Y)$ is *accepted* if there exists a computation connecting this word and W_0 .

3.2 Modifying the rules of S -machines

We shall need the following properties of our S -machines. All S -machines that appears in this paper, except for M_2 , satisfy Property 3.1 (1) below, and \tilde{M}_2 satisfies Property 3.1 (2).

Property 3.1. (1) *In every part $v_i q_i u_{i+1} \rightarrow v'_i q'_i u'_{i+1}$ we have $\|v_{i-1}\| + \|v'_{i-1}\| \leq 1$ and $\|u_i\| + \|u'_i\| \leq 1$ (see the notation in (**) above).*

(2) *For every rule, we have $\sum_i (\|v_i\| + \|v'_i\| + \|u_i\| + \|u'_i\|) \leq 1$.*

Suppose that Property 3.1 (1) is not satisfied. For example, suppose that a positive rule θ of an S -machine \mathcal{S} has the i -th part of the form $v_{i-1} a q_i u_i \rightarrow v'_{i-1} q'_i u'_i$, where $u_{i-1}, v_i, u'_{i-1}, v'_i$ are words in the appropriate parts of the alphabet of a -letters, v_{i-1} is not empty, a is a a -letter, q_i, q'_i are q -letters (a very similar procedure can be done in all other cases). Then we create a new S -machine $\tilde{\mathcal{S}}$ with the same standard base and the same a -letters as \mathcal{S} . In order to make $\tilde{\mathcal{S}}$, we add one new (*auxiliary*) q -letter \tilde{q}_i to each part of the set of q -letters, and replace the rule θ by two rules θ' and θ'' . The rule θ' is obtained from θ by replacing the part $v_{i-1} a q_i u_i \rightarrow v'_{i-1} q'_i u'_i$ by $a q_i u_i \rightarrow \tilde{q}_i u'_i$, and all other parts $U_j \rightarrow V_j$ by $U_j \rightarrow \tilde{q}_j$ (here \tilde{q}_j is the auxiliary q -letter in the corresponding part of the set of q -letters). The rule θ'' is obtained from θ by replacing the part $v_{i-1} a q_i u_i \rightarrow v'_{i-1} q'_i u'_i$ by $v_{i-1} \tilde{q}_i \rightarrow v'_{i-1} q'_i$, and all other parts $U_j \rightarrow V_j$ by $\tilde{q}_j \rightarrow V_j$.

The key property of the new S -machine is in the following obvious lemma.

Lemma 3.2. *There is a one-to-one correspondence between computations $w_0 \rightarrow \dots \rightarrow w_n$ of $\tilde{\mathcal{S}}$ (with any base) such that w_0, w_n do not have auxiliary q -letters and computations*

of \mathcal{S} connecting the same words. For every history H of such computation of \mathcal{S} , the corresponding history of computation of $\tilde{\mathcal{S}}$ is obtained from H by replacing every occurrence of the rule θ by the 2-letter word $\theta'\theta''$.

Note that the sum of lengths of words in all parts of θ' (resp. θ'') in $\tilde{\mathcal{S}}$ is smaller than the similar sum for θ . Clearly, applying this transformation to an \mathcal{S} -machine \mathcal{S} several times, we obtain a new \mathcal{S} -machine satisfying Property 3.1 (1). Similarly, one can obtain Property 3.1 (2). Thus Lemma 3.2 implies the following

Lemma 3.3. *For every \mathcal{S} -machine \mathcal{S} there exists an \mathcal{S} -machine $\tilde{\mathcal{S}}$ with the same standard base, the same set of a -letters, and some new, auxiliary, q -letters such that $\tilde{\mathcal{S}}$ satisfies Property 3.1 (2), and there exists a one-to-one correspondence between computations $w_0 \rightarrow \dots \rightarrow w_n$ of $\tilde{\mathcal{S}}$ (with any base) such that w_0, w_n do not have auxiliary q -letters and computations of \mathcal{S} connecting the same words. For every history H of \mathcal{S} , the corresponding history of computation of $\tilde{\mathcal{S}}$ is obtained from H by replacing every occurrence of the rule θ by the word $\phi(\theta)$ such that all rules in $\phi(\theta)$ are different, and $\phi(\theta), \phi(\theta')$ do not have common rules provided $\theta \neq \theta'$.*

3.3 Some general properties of \mathcal{S} -machines

Note that the base of an admissible word is not always a reduced word. But we have the following immediate corollary of the definition of admissible word.

Lemma 3.4. *If the i -th component of the rule θ has the form $q_i \xrightarrow{\ell} q'_i$, i.e. $Y_{i+1}(\theta) = \emptyset$, then the base of any admissible for θ word cannot have subwords $Q_i Q_{i+1}^{-1}$ or $Q_{i+1}^{-1} Q_{i+1}$.*

In this paper we are often using copies of words. If W is a word and A is an alphabet, then to obtain a *copy* of W in the alphabet A we substitute letters from A for letters in W so that different letters from A substitute for different letters. Note that if U' and V' are copies of U and V respectively corresponding to the same substitution, and $U' \equiv V'$, then $U \equiv V$.

The following lemma is obvious.

Lemma 3.5. *Suppose that the base of an admissible word W is $Q_i Q_{i+1}$. Suppose that each rule of a reduced computation starting with $W \equiv q_i u q_{i+1}$ and ending with $W' \equiv q'_i u' q'_{i+1}$ multiplies the $Q_i Q_{i+1}$ -sector by a letter on the left (resp. right). And suppose that different rules multiply that sector by different letters. Then the history of computation is a copy of the reduced form of the word $u' u^{-1}$ read from right to left (resp. of the word $u^{-1} u'$ read from left to right). In particular, if $u \equiv u'$, then the computation is empty.*

Lemma 3.6. *Let $W_0 \rightarrow \dots \rightarrow W_t$ be a sequence of transformations of reduced words, where W_i is a conjugate of W_{i-1} ($i = 1, \dots, t$) by a letter, and H - a product of these letters, i.e. $W_t = H^{-1} W_0 H$. Then H is equal to a reduced product $H_1 H_2^k H_3$, where $k \geq 0, \|H_2\| \leq \min(\|W_0\|, \|W_t\|)$, $\|H_1\| \leq \|W_0\|/2$, and $\|H_3\| \leq \|W_t\|/2$.*

Proof. One may assume that the consecutive transformations $W_{i-1} \rightarrow W_i \rightarrow W_{i+1}$ are not mutual inverse. If $\|W_{i-1}\| < \|W_i\|$, then $\|W_i\| - \|W_{i-1}\| = 2$, and moreover, $\|W_{i'}\| - \|W_{i'-1}\| = 2$, for every $i' \geq i$. Similar observation is true for inverse transformations $W_t \rightarrow \dots \rightarrow W_0$. It follows that there exist subscripts i, j such that $0 \leq i \leq j \leq t$, and each of the transformations $W_0 \rightarrow \dots \rightarrow W_i$ decreases the length by 2, while $\|W_i\| = \|W_{i+1}\| = \dots = \|W_j\|$, and each of the transformations $W_j \rightarrow \dots \rightarrow W_t$ increases the

length by 2. Thus, we have $W_i = (H'_1)^{-1}W_0H'_1$, where $\|W_0\| - \|W_i\| = 2\|H'_1\|$ and $W_t = (H'_3)^{-1}W_jH'_3$, where $\|W_t\| - \|W_j\| = 2\|H'_3\|$.

We also have $j - i$ one-letter cyclic shifts $W_i \rightarrow \dots \rightarrow W_j$, and this procedure is periodic with period $\|W_i\|$, whence the middle part of the conjugating word H must be of the form $\bar{H}_2^k \bar{H}$, with $k \geq 0$ and $\|\bar{H}\| < \|\bar{H}_2\| = \|W_i\| \leq \min(\|W_0\|, \|W_t\|)$, where \bar{H} is a prefix of the word \bar{H}_2 . Replacing \bar{H}_2 by a cyclic permutation H_2 one rewrites the same middle part as $H'H_2^kH''$, where $\|H_2\| = \|\bar{H}_2\|$ and $\|H'\|, \|H''\| \leq \frac{1}{2}(\|\bar{H}\| + 1) \leq \|W_i\|/2$. Finally, we set $H_1 \equiv H'_1H'$ and $H_3 \equiv H''H'_3$, to obtain the required factorization of H with $\|H_1\| \leq \frac{1}{2}(\|W_0\| - \|W_i\|) + \|W_i\|/2 = \|W_0\|/2$ and also $\|H_3\| \leq \|W_t\|/2$. \square

Lemma 3.6 immediately implies

Lemma 3.7. *Suppose that the base of an admissible word W is $Q_iQ_i^{-1}$ (resp., $Q_i^{-1}Q_i$). Suppose that each rule θ of a reduced computation starting with $W \equiv q_iuq_i^{-1}$ (resp., $q_i^{-1}uq_i$), where $u \neq 1$, and ending with $W' \equiv q'_iu'(q'_i)^{-1}$ (resp., $W' \equiv (q'_i)^{-1}u'q'_i$) has a component $q_i \rightarrow a_\theta q'_i b_\theta$, where b_θ (resp., a_θ) is a letter, and for different θ -s the b_θ -s (resp., a_θ -s) are different. Then the history of the computation has the form $H_1H_2^kH_3$, where $k \geq 0$, $\|H_2\| \leq \min(\|u\|, \|u'\|)$, $\|H_1\| \leq \|u\|/2$, and $\|H_3\| \leq \|u'\|/2$.*

3.4 Turing machines as S -machines

Every symmetric Turing machine M satisfying Condition (d) of Lemma 2.2 can be viewed as an S -machine SM [19, Page 372], such that the positive (negative) commands of M are the positive (negative) rules of SM . More precisely, we consider the α and ω symbols as state letters (hence the set of state letters has 2 more parts for each tape of the Turing machine). A part of a command of the form $uq_iv \rightarrow u'_iq'_iv'_i$, where u and v are tape letters or empty words, is replaced by

$$[\alpha_i \rightarrow \alpha_i, u_iq_iv_i \rightarrow u'_iq'_iv'_i, \omega_i \xrightarrow{\ell} \omega_i],$$

a part of the form $\alpha_iq_iv_i \rightarrow \alpha_iq'_iv'_i$ is replaced by

$$[\alpha_i \xrightarrow{\ell} \alpha_i, q_iv_i \rightarrow q'_iv'_i, \omega_i \xrightarrow{\ell} \omega_i],$$

a part of the form $u_iq_i\omega_i \rightarrow u'_iq'_i\omega_i$ is replaced by

$$[\alpha_i \rightarrow \alpha_i, u_iq_i \xrightarrow{\ell} u'_iq'_i, \omega_i \xrightarrow{\ell} \omega_i],$$

and a part of the form $\alpha_iq_i\omega_i \rightarrow \alpha_iq'_i\omega_i$ is replaced by

$$[\alpha_i \xrightarrow{\ell} \alpha_i, q_i \xrightarrow{\ell} q'_i, \omega_i \xrightarrow{\ell} \omega_i].$$

The language recognized by SM is in general much bigger than the language recognized by M since M works with a *positive* tape alphabet only. Nevertheless the following property statement holds:

Lemma 3.8. *(compare with [19, Proposition 4.1]) Let M be a symmetric Turing machine satisfying the conditions of Lemma 2.3 (i.e. the symmetrization of some deterministic Turing machine satisfying conditions of Lemma 2.2). Let $W_1 \rightarrow W_2 \rightarrow \dots \rightarrow W_k$ be a computation of the S -machine SM with the standard base consisting of positive words. Then it is a computation of the Turing machine M (with the same history).*

Proof. Indeed, every positive admissible word W of SM with the standard base is a configuration of the Turing machine M . If a rule θ of SM satisfying Property (c) of Lemma 2.2 or its inverse applies to this W and the word $W \cdot \theta$ is positive, then, obviously, the command θ of M applies to W and the result of the application is the same (here we essentially use the fact that the rule θ or its inverse inserts (deletes) at most one letter). This immediately implies the statement of the lemma. \square

4 The S -machine

We turn to the proof of Theorem 1.1. From now M_0 is the deterministic Turing machine recognizing language X from Theorem 14.1, the machine M_1 is constructed as in Lemma 2.2 and recognizes the language X_1 , where $X_1 = X$. We keep the same notation M_1 for the symmetrization of M_1 given by Lemma 2.3. Note that by claim (e) of that lemma, the machine M_1 has infinitely many h_α -good numbers for every $\alpha > 0$, where the functions h_α are defined in Theorem 14.1. First, we need to construct a new S -machine which inherits important properties of the Turing machine M_1 .

As in Section 3.3, we can view M_1 as an S -machine. We shall denote that S -machine by the same letter M_1 .

Let $Q_0 \dots Q_N$ be the standard base of M_1 , let the components of the alphabet of a -letters be Y_1, \dots, Y_N (letters from Y_i are in the $Q_{i-1}Q_i$ -sectors of admissible words with the standard base).

4.1 The machine $M_1 \circ Z$

Let A be a finite set of letters. Let the sets A_1, A_2 be copies of A . It will be convenient to denote A by A_0 . For every letter $a_0 \in A_0$ let a_1, a_2 denote its copies in A_1, A_2 .

As in [15], consider the following auxiliary "adding" S -machine $Z(A)$.

Its set of state letters is $P_1 \cup P_2 \cup P_3$, where

$$P_1 = \{L\}, P_2 = \{p(1), p(2), p(3)\}, P_3 = \{R\}.$$

The set of tape letters is $Y_1 \cup Y_2$, where $Y_1 = A_0 \cup A_1$ and $Y_2 = A_2$.

The machine $Z(A)$ has the following positive rules (there a is an arbitrary letter from A). The comments explain the meanings of these rules.

- $r_1(a) = [L \rightarrow L, p(1) \rightarrow a_1^{-1}p(1)a_2, R \rightarrow R]$.

Comment. The state letter $p(1)$ moves left searching for a letter from A_0 and replacing letters from A_1 by their copies in A_2 .

- $r_{12}(a) = [L \rightarrow L, p(1) \rightarrow a_0^{-1}a_1p(2), R \rightarrow R]$.

Comment. When the first letter a_0 of A_0 is found, it is replaced by a_1 , and $p(1)$ turns into $p(2)$.

- $r_2(a) = [L \rightarrow L, p(2) \rightarrow a_0p(2)a_2^{-1}, R \rightarrow R]$.

Comment. The state letter $p(2)$ moves toward R .

- $r_{21} = [L \rightarrow L, p(2) \xrightarrow{\ell} p(1), R \rightarrow R], Y_1(r_{21}) = Y_1, Y_2(r_{21}) = \emptyset$.

Comment. $p(2)$ and R meet, the cycle starts again.

- $r_{13} = [L \xrightarrow{\ell} L, p(1) \rightarrow p(3), R \rightarrow R], Y_1(r_{13}) = \emptyset, Y_2(r_{13}) = A_2$.

Comment. If $p(1)$ never finds a letter from A_0 , the cycle ends, $p(1)$ turns into $p(3)$; p and L must stay next to each other in order for this rule to be executable.

- $r_3(a) = [L \rightarrow L, p(3) \rightarrow a_0 p(3) a_2^{-1}, R \rightarrow R], Y_1(r_3(a)) = A_0, Y_2(r_3(a)) = A_2$

Comment. The letter $p(3)$ returns to R .

For every letter $a \in A$ we set $r_i(a^{-1}) = r_i(a)^{-1}$ ($i = 1, 2, 3$).

The following Lemmas from [15] contain the main properties of $Z(A)$ used later.

If $u \equiv a_1 \dots a_m$ is a word, a_i are letters, then we set $r_3(u) \equiv r_3(a_1) r_3(a_2) \dots r_3(a_m)$, $r_2(u) \equiv r_2(a_1) r_2(a_2) \dots r_2(a_m)$, $r_1(u) \equiv r_1(a_m) \dots r_1(a_2) r_1(a_1)$.

Lemma 4.1. ([15, Lemma 3.18]) Suppose that an admissible word W of $Z(A)$ has the form $LupvR$, where u, v are words in $(A_0 \cup A_1 \cup A_2)^{\pm 1}$. Let $W \cdot \theta \equiv Lu'p'v'R$. Then the projections of uv and $u'v'$ onto A are freely equal.

Lemma 4.2. (Follows from the proof of [15, Lemma 3.25]) Let W be an admissible word of $Z(A)$ with $\text{base}(W) \equiv LpR$. Then for every reduced computation $W \equiv W_0 \rightarrow W_1 \rightarrow \dots \rightarrow W_t \equiv W \cdot H$ of the S -machine $Z(A)$:

1. $\|W_i\| \leq \max(\|W_0\|, \|W \cdot H\|)$, $i = 0, \dots, t$,
2. If $W \equiv LupR$, where $p = p(1)$ (resp. $p = p(3)$), u is positive, then there exists a computation starting with W and ending with $Lup(3)R$ (resp. $Lup(1)R$). Moreover if u is any word in a -letters, and for some history of computation H , $W \cdot H$ contains $p(3)R$ (resp. $p(1)R$) and all a -letters in $W, W \cdot H$ are from $A_0^{\pm 1}$, then the length of H is between $2^{\|u\|}$ and $6 \cdot 2^{\|u\|}$, u and all words in the computation $W \rightarrow \dots \rightarrow W \cdot H$ are positive, all words in that computation have the same length, and H is uniquely determined by u . That computation (resp. its inverse) has the history of the following form

$$D(u) \equiv E(u) r_{13} r_3(u),$$

where $E(u)$ is defined by induction: $E(\emptyset) \equiv \emptyset$ and if $u \equiv a_1 u'$, then

$$E(u) \equiv E(u') r_{12}(a_1) r_2(u') E(u') r_1(a_1)$$

Lemma 4.3. (The first part of the lemma is [15, Lemma 3.21]) For every admissible word W of $Z(A)$ with $\text{base}(W) \equiv LpR$, every rule θ applicable to W , and every natural number $t > 1$, there is at most one reduced computation $W \rightarrow_{\theta} W_1 \rightarrow \dots \rightarrow W_t$ of length t , where the lengths of the words are all the same. (In fact from the proof of [15, Lemma 3.21], it immediately follows that the history of that computation is a subword of $D(u)^{\pm 1}$ for some u).

Lemma 4.4. ([15, Lemma 3.27]) Suppose W is an admissible word of $Z(A)$ with $\text{base}(W) \equiv LpR$. Suppose that $W \cdot H$ exists for some reduced history H . Suppose that both W and $W \cdot H$ contain $p(1)R$ (resp. $p(3)R$) and all a -letters in $W, W \cdot H$ are from A_0 . Then H is empty.

Lemma 4.5. ([15, Lemma 3.24]) Let $W \equiv LvpuR$ and $\text{base}(W) \equiv LpR$. Suppose that $\|W \cdot \theta\| > \|W\|$. Then for every reduced computation $W \rightarrow_\theta W_1 \rightarrow W_2 \rightarrow \dots \rightarrow W \cdot H$, we have $\|W_i\| > \|W\|$ for every $i \geq 1$.

We define the S-machine M_2 as the composition $M_1 \circ Z$ (this operation is defined in [15], see also below).

Essentially we insert a p -letter between any two consecutive q -letters in admissible words of M_1 with standard base, and treat any subword $q_i \dots p_{i+1} \dots q_{i+1}$ as an admissible word for $Z(A)$ (that is q_i plays the role of L and q_{i+1} plays the role of R). The only differences with the construction in [15] are that, for every state letter q , we keep the sets of a -letters that can appear to the left and to the right of q disjoint, and after application of a main rule, not only the state letters of copies of $Z(A)$ remember the main rule but also the state letters coming from M_1 remember that rule. These changes do not affect the proofs of statements in [15] that we are going to use.

Let us describe M_2 in details. First, for every $i = 1, \dots, N$, we make three copies of the alphabet Y_i of M_1 ($i = 1, \dots, N$): $Y_{i,0} = Y_i$, $Y_{i,1}$, $Y_{i,2}$. Let Θ be the set of positive commands of M_1 (viewed as rules of an S-machine). The set of state letters of the new machine is

$$Q'_0 \cup P_1 \cup Q'_1 \cup P_2 \cup \dots \cup P_N \cup Q'_N,$$

where $P_i = \{p^{(i)}, p^{(1,i)}, p^{(0,i)}, p_1^{(\theta,i)}, p_2^{(\theta,i)}, p_3^{(\theta,i)} \mid \theta \in \Theta\}$ $i = 1, \dots, N$, $Q'_i = Q_i \sqcup (Q_i \times \Theta)$, where $Q_0 \sqcup Q_1 \sqcup \dots \sqcup Q_N$ is the set of state letters of M_1 . We shall denote a pair (q, θ) from $Q_i \times \Theta$ in Q'_i by $q^{(\theta,i)}$. Thus every "old" state letter of M_1 gets "multiple" copies indexed by positive rules of M_1 , and the state letters of various copies of Z have upper indices corresponding to the positive rules of M_1 and the number of sector where the machine is inserted. The $Q'_0 P_1$ -sector of an admissible word with the standard base will be called the *input* sector of that word.

The set of tape letters is

$$\bar{Y} = (Y_{1,0} \sqcup Y_{1,1}) \sqcup Y_{1,2} \sqcup (Y_{2,0} \sqcup Y_{2,1}) \sqcup Y_{2,2} \sqcup \dots \sqcup (Y_{N,0} \sqcup Y_{N,1}) \sqcup Y_{N,2};$$

the components of this union will be denoted by $\bar{Y}_1, \dots, \bar{Y}_{2N}$. We shall call the new (second) indices of tape letters the M_2 -indices of these letters.

The set of positive rules $\bar{\Theta}$ of $M_1 \circ Z$ is a union of the set of suitably modified positive rules of M_1 and $|\Theta|N$ copies $Z^{(\theta,i)}$ ($\theta \in \Theta, i = 1, \dots, N$) of positive rules of the machine $Z(Y_i)$ (also suitably modified).

More precisely, every rule $\theta \in \Theta$ of the form

$$[q_0 u_1 \rightarrow q'_0 u'_1, v_1 q_1 u_2 \rightarrow v'_1 q'_1 u'_2, \dots, v_N q_N \rightarrow v'_N q'_N],$$

where $q_i, q'_i \in Q_i$, u_i and v_i are words in Y , is replaced by

$$\bar{\theta} = \left[\begin{array}{l} q_0 u_1 \rightarrow (q')^{(\theta,0)} u'_1, v_1 p^{(1)} \xrightarrow{\ell} v'_1 p_1^{(\theta,1)}, q_1 u_2 \rightarrow (q')^{(\theta,1)} u'_2, \dots, \\ v_N p^{(N)} \xrightarrow{\ell} v'_N p_1^{(\theta,N)}, q_N \rightarrow (q')^{(\theta,N)} \end{array} \right]$$

with $\bar{Y}_{2i-1}(\bar{\theta}) = Y_{i,0}(\theta)$ and $\bar{Y}_{2i}(\bar{\theta}) = \emptyset$. If $\theta = \theta_{start}$ is the unique start rule of M_1 then all $p^{(i)}$ -s in the above definition of $\bar{\theta}$ must be replaced by the special start letters $p^{(1,i)}$ -s.

Thus each modified rule from $\bar{\Theta}$ turns on N copies of the machine $Z(A)$ (for different A 's). The rule $\bar{\theta}$ will be called *the rule of M_2 corresponding to the rule θ of M_1* .

Each machine $Z^{(\theta,i)}$ is a copy of the machine $Z(Y_i(\theta))$, where every rule $\tau = [U_1 \rightarrow V_1, U_2 \rightarrow V_2, U_3 \rightarrow V_3]$ is replaced by the rule of the form

$$\bar{\tau}_i(\theta) = \left[\begin{array}{l} \bar{U}_1 \rightarrow \bar{V}_1, \bar{U}_2 \rightarrow \bar{V}_2, \bar{U}_3 \rightarrow \bar{V}_3, \\ (q')^{(\theta,j)} \rightarrow (q')^{(\theta,j)}, p_3^{(\theta,j)} \xrightarrow{\ell} p_3^{(\theta,j)}, j = 1, \dots, i-1, \\ p_1^{(\theta,s)} \xrightarrow{\ell} p_1^{(\theta,s)}, (q')^{(\theta,s+1)} \rightarrow (q')^{(\theta,s+1)}, s = i+1, \dots, N-1 \end{array} \right],$$

where $\bar{U}_1, \bar{U}_2, \bar{U}_3, \bar{V}_1, \bar{V}_2, \bar{V}_3$ are obtained from $U_1, U_2, U_3, V_1, V_2, V_3$, respectively, by replacing $p(j)$ with $p_j^{(\theta,i)}$, L with $(q')^{(\theta,i)}$, and R with $(q')^{(\theta,i+1)}$, and for $s \neq i$, $\bar{Y}_{2s-1}(\bar{\tau}_i(\theta)) = Y_{s,0}(\theta)$. Thus while the machine $Z^{(\theta,i)}$ works all other machines $Z^{(\theta,j)}$, $j \neq i$ must stay idle (their state letters do not change and do not move away from the corresponding q -letters). After the machine $Z^{(\theta,i)}$ finishes (i.e. the state letter $p_3^{(\theta,i)}$ appears next to $(q')^{(\theta,i)}$), the next machine $Z^{(\theta,i+1)}$ starts working.

In addition, we need the following *transition* rule $\zeta(\theta)$ that removes θ from all state letters, and turns all $p_3^{(\theta,j)}$ back into $p^{(j)}$:

$$[(q')^{(\theta,i)} \rightarrow q'_i, p_3^{(\theta,j)} \xrightarrow{\ell} p^{(j)}, i = 0, \dots, N, j = 1, \dots, N].$$

If θ is the unique accept rule of M_1 then all $p^{(j)}$ -s in the above definition of $\zeta(\theta)$ must be replaced by the special (accept) letters $p^{(0,i)}$ -s.

Lemma 4.6. *Let H be the history of a reduced computation $W \rightarrow \dots \rightarrow W \cdot H$ of M_2 with the standard base, of the form $\bar{\theta}H'\zeta(\theta)$, where θ is a positive rule of the S -machine M_1 , H' does not contain rules corresponding to rules of M_1 and occurrences of $\zeta(\theta)^{\pm 1}$. Let $W \cdot \bar{\theta} = q^{(\theta,0)}u_1p^{(\theta,1)}q^{(\theta,1)}u_2\dots u_Np^{(\theta,N)}q^{(\theta,N)}$ for some words u_i in $Y_{i,0}$. Then $H' \equiv H_1H_2\dots H_N$, where each H_s is the computation of the machine $Z^{(\theta,s)}$ whose history is a copy of $D(u_s)$ (described in Lemma 4.2, part 2), and all words in the computation $W \cdot \bar{\theta} \rightarrow \dots \rightarrow W \cdot H$ are positive. We shall denote H by $\Pi_{1,2}(\theta, W)$. That computation is completely determined by its first (last) word and θ .*

Proof. By assumption, H' consists of the rules of various S -machines $Z^{(\theta,i)}$ since only rules $\bar{\theta}^{-1}$ and $\zeta(\theta)$ can remove the θ -index of state letters. Since the word $W' = W \cdot (\bar{\theta}H')$ is in the domain of $\zeta(\theta)$, all p -letters in W' have the form $p_3^{(\theta,i)}$. Since rules from $Z^{(\theta,i)}$ can apply to an admissible word with the standard base only if p_j -letters ($j \neq i$) stay next to the left of the corresponding (copies of) state letters of M_1 , and have the form $p_3^{(\theta,j)}$, if $j < i$, and the form $p_1^{(\theta,j)}$ if $j > i$, we can conclude that $H' \equiv H_1H_2\dots H_m$, where each H_s is a non-empty history of the computation of some $Z^{(\theta,j(s))}$ such that $j(s+1) = j(s) \pm 1$, $j(1) = 1$, and each m between 1 and N occurs as $j(s)$ for some s . Note that if $j(s+1) = j(s) - 1$, then there must be $s' > 1$ such that $j(s'-1) = j(s'+1) = j(s') - 1$. But then the subcomputation of the S -machine $Z^{(\theta,j(s))}$ with history H_s starts and ends with the p -letter of the form $p_1^{(\theta,j(s))}$. By Lemma 4.4, then H_s is empty, a contradiction. Hence $j(s+1) = j(s) + 1$ for every s , which implies that $m = N$, and $j(s) = s$ for every s . By Lemma 4.2, part 2, each H_s is uniquely determined by the word u_s (and θ) and is equal to a copy of $D(u_s)$ defined in Lemma 4.2, part 2. This implies the uniqueness of H' . The fact that all words in the computation $W \cdot \bar{\theta} \rightarrow \dots \rightarrow W \cdot H$ are positive follows from Lemma 4.2, part 2. \square

Lemma 4.7. *Let H be the history of a reduced computation $W \rightarrow \dots \rightarrow W \cdot H$ of M_2 with the standard base, $H \equiv \zeta(\theta_1)^{e_1}H'\zeta(\theta_2)^{e_2}$, where θ_1, θ_2 are positive rules of the S -machine*

M_1 , H' does not contain rules corresponding to rules of M_1 and occurrences of $\zeta(\theta)^{\pm 1}$, $\epsilon_1, \epsilon_2 = \pm 1, i = 1, 2$. Then H' is empty, $\epsilon_1 = 1$, and $\epsilon_2 = -1$.

Proof. At first, let us prove that H' is empty. If $\epsilon_1 = 1$, then the p -letters in $W \cdot \zeta(\theta_1)$ have the form $p^{(j)}$, and no rules from any $Z^{(\theta, s)}$ apply to words with such p -letters, hence H' is empty (because by assumption it can contain only rules of various $Z^{(\theta, j)}$). Thus we can assume that $\epsilon_1 = -1$. Similarly $\epsilon_2 = 1$. As in the proof of Lemma 4.6, $H' \equiv H_1 H_2 \dots H_m$, where each H_i is a non-empty history of computation of some $Z^{(\theta_1, j(s))}$ such that $j(s+1) = j(s) \pm 1$, $j(1) = N$. Note that the p -letters in $W \cdot \zeta(\theta_1)^{-1}$ and in $W \cdot \zeta(\theta_1)^{-1} H'$ are of the form $p_3^{(\theta_1, j)}$. Therefore for some s , we must have $j(s-1) = j(s+1) = j(s) + 1$. As in the proof of Lemma 4.6, this implies that H_s is empty, a contradiction.

Since H' is empty, the p -letters in $W \cdot \zeta(\theta_1)$ have no θ -indices because otherwise they have to be equal to both θ_1 and θ_2 , and $H \equiv \zeta(\theta_1)^{-1} \zeta(\theta_1)$ would not be reduced. Therefore $\epsilon_1 = 1$, and $\epsilon_2 = -1$. \square

Lemma 4.8. *Let H be the history of a reduced computation $W \rightarrow \dots \rightarrow W \cdot H$ of M_2 with the standard base, $H \equiv \bar{\theta}_1^{-1} H' \bar{\theta}_2$, where θ_1, θ_2 are positive rules of the S -machine M_1 , H' does not contain rules corresponding to the rules of M_1 . Then H' is empty.*

Proof. Indeed if H' is not empty, it must start with $\zeta(\theta_3)^{-1}$, and end with $\zeta(\theta_4)$, for some positive θ_3, θ_4 , and then H' would have a subword of the form $\zeta(\theta')^{-1} H'' \zeta(\theta'')$ satisfying to the assumptions of Lemma 4.7, which contradicts with the statement of Lemma 4.7. \square

Lemma 4.9. *Let H be the history of a reduced non-empty computation $W \rightarrow \dots \rightarrow W \cdot H$ of M_2 with the standard base, $H \equiv \bar{\theta}_1 H' \bar{\theta}_2^{-1}$, where θ_1, θ_2 are positive rules of the S -machine M_1 , and let H' contain no rules corresponding to rules of M_1 . Then $H \equiv \Pi_{1,2}(\theta_1, W) \Pi_{1,2}(\theta_2, W \cdot H)^{-1}$, all words in that computation except possibly the first and the last ones are positive, and $\theta_1 \neq \theta_2$.*

Proof. If H does not contains the rule $\zeta(\theta_1)$, we get that H' is empty as in the proof of Lemma 4.7. Note that the p -letters of the admissible words in the domain of $\bar{\theta}_1^{-1}$ (of $\bar{\theta}_2^{-1}$) have the form $p_1^{(\theta_1, i)}$ (resp., $p_1^{(\theta_2, i)}$). It follows that $\theta_1 \equiv \theta_2$, and so the history H is not reduced, a contradiction.

Hence we can assume that H contains $\zeta(\theta_1)$. Then the next rule in H must be $\zeta(\theta_3)^{-1}$ for some θ_3 because only rules of the form $\bar{\theta}$ and $\zeta(\theta)^{-1}$ for positive θ are applicable to admissible words in the range of $\zeta(\theta_1)$ and H' does not contain rules corresponding to rules of M_1 . After application of $\zeta(\theta_3)^{-1}$, all p -letters in the admissible word have the form $p_3^{(\theta_3, i)}$. Recall that the p -letters of the admissible words in the domain of $\bar{\theta}_2^{-1}$ have the form $p_1^{(\theta_2, i)}$. Thus if $\theta_3 \neq \theta_2$, the word H' must contain a subword $\zeta(\theta_3)^{-1} H'' \zeta(\theta_3)$, where H'' does not contain rules of the form $\zeta(\theta)$ and their inverses. By Lemma 4.7, H'' is empty, and the computation is not reduced, a contradiction. Hence H' has the form $H'' \zeta(\theta_1) \zeta(\theta_2)^{-1} H'''$, where H'', H''' do not contain rules of the form $\bar{\theta}^{\pm 1}$ and rules of the form $\zeta(\theta)^{\pm 1}$ by Lemma 4.7. Applying now Lemma 4.6 to the computation with history $\bar{\theta}_1 H'' \zeta(\theta_1)$ and the first word W , and to the computation with history $\bar{\theta}_2 (H''')^{-1} \zeta(\theta_2)$ and the first word $W \cdot H$, we obtain the desired equality $H \equiv \Pi_{1,2}(\theta_1, W) \Pi_{1,2}(\theta_2, W \cdot H)^{-1}$ and the fact that all words in that computation are positive except possibly the first and the last words. Now if $\theta_1 = \theta_2$, then H' must be empty (since the computation is

reduced and H' is a product of two mutually inverse words in that case by the uniqueness statement of Lemma 4.6) (b), and so the computation is empty, a contradiction. \square

For every admissible word W of M_2 with the standard base, let $\pi_{2,1}(W)$ be the word obtained by removing state p -letters, removing θ -indices (if any exist) of other state letters, and removing the M_2 -indices of a -letters, and reducing the resulting word. We obtain a word in the alphabet of state and tape letters of M_1 .

With every admissible word W of the S -machine M_1 with the standard base we associate the admissible word $\pi_{1,2}(W)$ of M_2 by inserting the state letters $p^{(j)}$ -s next to the left of q_j -s, and replacing every a -letter a by a_0 . (We insert the special letters $p^{(1,j)}$ -s instead of $p^{(j)}$ -s if the word W is admissible for the unique start command of M_1 .) Let W_0 be the stop word of M_1 (considered as an S -machine). It exists because the Turing machine M_1 is recognizing. We call the word $\pi_{1,2}(W_0)$ the *stop word* of M_2 .

Note that we have

$$\pi_{2,1}(\pi_{1,2}(W)) \equiv W. \quad (4.2)$$

For every input configuration W of the S -machine M_1 , we call $\pi_{1,2}(W)$ an *input word* of M_2 . Note that an input word of M_2 has the standard base and all sectors except the q_0p_1 -sector are empty.

For every rule θ' of M_2 , if $(\theta')^{\pm 1}$ corresponds to a positive rule θ (i.e. if $\theta' \equiv \bar{\theta}^{\pm 1}$) of M_1 we denote $\theta^{\pm 1} = \Pi_{2,1}(\theta')$. If $(\theta')^{\pm 1}$ does not correspond to a rule of M_1 , we denote by $\Pi_{2,1}(\theta')$ the empty rule. The map $\Pi_{2,1}$ extends to histories of computations in the natural way.

Lemma 4.10. *If H is the reduced history of a computation of M_2 with the standard base and $W \cdot H = W'$, then $\Pi_{2,1}(H)$ is a reduced history of computation of the S -machine M_1 . If $\pi_{2,1}(W)$ is an admissible word for the S -machine M_1 , then*

$$\pi_{2,1}(W) \cdot \Pi_{2,1}(H) = \pi_{2,1}(W'). \quad (4.3)$$

Proof. The fact that $\pi_{2,1}(H)$ is a history of computation of the S -machine M_1 , and (4.3), immediately follows from the definition of M_2 and the fact that application of rules from $Z^{(\theta,i)}$ does not change the value of $\pi_{2,1}$ by Lemma 4.1. The fact that $\pi_{2,1}(H)$ is reduced follows from Lemmas 4.9 and 4.8 \square

The next lemma-definition gives in a sense an inverse function of $\Pi_{2,1}$.

Lemma 4.11. *For every positive $\bar{\theta}$ -admissible word W of M_2 with the standard base such that there exists a computation $w \rightarrow w_1 \rightarrow \dots \rightarrow w \cdot H$ of the Turing machine M_1 with positive history H , starting with $w \equiv \pi_{2,1}(W)$ and having the first rule θ , and all admissible words positive there exists a unique reduced computation of M_2 starting with $W \rightarrow W \cdot \bar{\theta}$, whose history is H' such that $\Pi_{2,1}(H') \equiv H$ and the last rule is of the form $\zeta(\theta)$. That history H' will be denoted by $\Pi_{1,2}(H, W)$. This definition agrees with the notation $\Pi_{1,2}(\theta, W)$ of Lemma 4.6.*

Proof. Indeed, if $H \equiv \theta\theta_1\dots\theta_n$, where all θ_i are positive rules of M_1 , then we can define $\Pi_{1,2}(H, w)$ as $\Pi_{1,2}(\theta, W)\Pi_{1,2}(\theta_1, W_1)\dots\Pi_{1,2}(\theta_n, W_n)$, where $W_i = \pi_{1,2}(w_i)$ ($i = 1, \dots, n$)

For the uniqueness of H' , we note that every rule of the form $\bar{\theta}_i$ can follow only after a rule of the form $\zeta(*)$ since H is positive. It follows from Lemma 4.7 that there is only

one $\zeta(*)^{\pm 1}$ -rule between $\bar{\theta}_i$ and a preceding rule of this form. Now the uniqueness of H' follows from Lemmas 4.6. \square

Every time we are using the notation $\Pi_{1,2}(H, W)$ below, the conditions of Lemma 4.11 will be assumed or clearly satisfied.

Remark 4.12. Note that if $\pi_{2,1}(W) \cdot H = W_0$, that is the computation of the Turing machine M_1 is accepting, then the corresponding computation of M_2 with history $\Pi_{1,2}(H, W)$ is also accepting.

Lemma 4.13. *There are no reduced computations $W \rightarrow W \cdot \bar{\theta}^{-1} \rightarrow W \cdot \bar{\theta}^{-1}\bar{\theta}'$ with the standard base, where the first and the third words are positive and θ, θ' are positive rules of M_1 .*

Proof. Assume that such a computation exists and $W \cdot \bar{\theta}^{-1}$ is positive too. Then by Lemma 3.8 this computation is a reduced computation of the symmetric Turing machine M_1 with history $\theta^{-1}\theta'$, contrary to the Property (b) of M_1 given by Lemma 2.3.

Now assume that the word $W \cdot \bar{\theta}^{-1}$ is not positive. By Property (b) from Lemma 2.2, each θ, θ' inserts/deletes at most one tape letter. The only non-trivial case is when θ, θ' insert an a -letter: in other cases the second word in the computation is obviously positive. But then $\bar{\theta}^{-1}$ must insert a letter a^{-1} which is then removed by $\bar{\theta}'$ (in the same sector). Since both rules $\bar{\theta}, \bar{\theta}'$ have word $W \cdot \bar{\theta}^{-1}$ in their domains, the left hand sides in all parts of the rules of θ, θ' coincide. Since M_1 is the symmetrization of a deterministic Turing machine by construction, $\theta \equiv \theta'$ and our computation is not reduced, a contradiction. \square

Lemma 4.14. *Let H be the history of a reduced computation $W \rightarrow \dots \rightarrow W \cdot H$ of M_2 with the standard base.*

(1) *If $H \equiv \bar{\theta}_1 H' \zeta(\theta_2)$, where θ_1, θ_2 are positive rules of the S -machine M_1 . Then the word H and all words in that computation except possibly the first one are positive.*

(2) *If $H \equiv \bar{\theta}_1 H_1 \bar{\theta}_2^{\epsilon_2} \dots \bar{\theta}_n^{\epsilon_n} H_n \bar{\theta}_{n+1}^{\epsilon_{n+1}}$, where $\theta_1, \dots, \theta_{n+1}$, are positive rules of the S -machine M_1 , $\epsilon_i = \pm 1$, $(\epsilon_n, \epsilon_{n+1}) \neq (-1, 1)$, and H_1, \dots, H_n have no rules corresponding to the rules of M_1 , then all words in this computation except possibly the first one and the last one are positive.*

Proof. (1) Induction on the length of $\Pi_{2,1}(H)$. Suppose $\Pi_{2,1}(H) \equiv \theta_1$. Then by Lemma 4.7, H' does not contain rules of the form $\zeta(\theta)^{\pm 1}$. Hence we can apply Lemma 4.6 and conclude that all words in the computation except possibly the first one are positive.

Suppose that the length of $\Pi_{2,1}(H)$ is at least 2. Suppose further that the second letter of $\Pi_{2,1}(H)$ is positive, that is $H \equiv \bar{\theta}_1 H_1 \bar{\theta}_3 H_2$ for some positive θ_3 and H_1 not containing rules corresponding to the positive rules of M_1 . Then H_1 must end with $\zeta(\theta')$ for some θ' . Then we can apply the induction assumption to the computations with histories $\bar{\theta}_1 H_1$ and $\bar{\theta}_3 H_2$ and conclude that H and all words, except for the first one, in the computation with history H are positive as desired.

Now suppose that the second letter in $\Pi_{2,1}(H)$ is θ_3^{-1} for some positive θ_3 . Since H ends with $\zeta(\theta_2)$, the last rule in $\Pi_{2,1}(H)$ is positive. Indeed the rule used in any reduced computation of M_2 immediately after a rule of the form $\bar{\theta}^{-1}$ for some positive θ is either $\zeta(\theta')^{-1}$ for some positive θ' or $\bar{\theta}'$ for some positive θ' (this can be seen by looking at the indices of q -letters of the admissible words). The first option is impossible by Lemma 4.7, the second option is impossible since we consider the last rule in $\Pi_{2,1}(H)$. Hence $H \equiv \bar{\theta}_1 H_1 \bar{\theta}_2^{-1} H_2 \dots H_{m-1} \bar{\theta}_m^{-1} H_m \bar{\theta}_{m+1} H''$ for some positive rules

$\theta_2, \dots, \theta_{m+1}$, where H_2, \dots, H_m do not contain rules corresponding to rules of M_1 or their inverses, θ_{m+1} is the second positive rule in $\Pi_{2,1}(H)$. By Lemma 4.8, H_m is empty. By Lemma 4.9, $\bar{\theta}_1 H_1 \bar{\theta}_2^{-1} \equiv \Pi_{1,2}(\theta_1, W) \Pi_{1,2}(\theta_2, W \cdot \bar{\theta}_1 H_1 \bar{\theta}_2^{-1})^{-1}$. Now, consider the computation of M_2 started with the admissible word $W' = W \cdot \bar{\theta}_1 H_1 \bar{\theta}_2^{-1} H_2 \dots H_{m-1} \bar{\theta}_m^{-1}$ and having the history $\bar{\theta}_m H_{m-1}^{-1} \dots H_2^{-1} \Pi_{1,2}(\theta_2, W \cdot \bar{\theta}_1 H_1 \bar{\theta}_2^{-1})$. By the inductive hypothesis, all words in this computation, starting with the second one, and all words in the computation $W \cdot \bar{\theta}_1 \rightarrow \dots \rightarrow W \cdot \Pi_{1,2}(\theta_1, W)$ are positive. By the induction assumption, the computation $W' \cdot \bar{\theta}_{m+1} \rightarrow \dots \rightarrow W \cdot H$ also consists of positive words. Therefore the first and the third words in the subcomputation with history $\bar{\theta}_m^{-1} \bar{\theta}_{m+1}$ are positive contrary to Lemma 4.13.

(2) It is nothing to prove if $n = 0$. The case $\epsilon_2 = 1$ was considered in the proof of claim (1). For the case $H \equiv \bar{\theta}_1 H_1 \bar{\theta}_2^{-1}$, we also proved that the computation with sub-history $H \equiv \bar{\theta}_1 H_1$ has all words positive except possibly the first one. Hence we may assume that $n \geq 2$. If $H \equiv \bar{\theta}_1 H_1 \bar{\theta}_2^{-1} H_2 \dots H_{n-1} \bar{\theta}_n^{-1} H_n \bar{\theta}_{n+1}^{-1}$, then we consider the computation with history H^{-1} and again come to the case $\epsilon_2 = 1$. Therefore we assume that $H \equiv \bar{\theta}_1 H_1 \bar{\theta}_2^{-1} H_2 \dots H_{m-1} \bar{\theta}_m^{-1} H_m \bar{\theta}_{m+1} H''$, where $m < n$ by the condition on $(\epsilon_n, \epsilon_{n+1})$. By Lemma 4.8, we have that H_m is empty. By the inductive hypothesis, the computations with histories $\bar{\theta}_1 H_1 \bar{\theta}_2^{-1} H_2 \dots H_{m-1} \bar{\theta}_m^{-1}$ and $\bar{\theta}_m H''$ have all words positive except possibly the first one and the last one. Therefore we can apply Lemma 4.13 to the computation with history $\bar{\theta}_m^{-1} \bar{\theta}_{m+1}$, a contradiction. \square

Lemma 4.15. *For every reduced computation $w_0 \rightarrow \dots \rightarrow w_t$ of M_2 with the standard base and a non-empty history H , we have $w_t \neq w_0$.*

Proof. Assume that $w_t = w_0$, and $t > 0$ is minimal. Then the computation $w_1 \rightarrow \dots \rightarrow w_{t-1}$ is not a counter-example, and so H is a cyclically reduced word.

If $H' \equiv \Pi_{21}(H)$ is empty, we consider the computation $w_0 \rightarrow \dots \rightarrow w_0 \circ H = w_0 \rightarrow \dots \rightarrow w_0 \circ H^2 \rightarrow \dots$ with history H^k , where k as large as we want. As in Lemmas 4.6 and 4.9, we have a decomposition $H \equiv H_1 \dots H_s$, where H_i corresponds to the work of some $Z^{(\theta,i)}$ or equal to some $\zeta(\theta)^{\pm 1}$ and s is bounded by a constant independent of k . It follows that H corresponds to only one $Z^{(\theta,i)}$, and then the equality $w_t = w_0$ and Lemma 4.5 imply that $\|w_0\| = \|w_1\| = \dots = \|w_t\|$. Now $\|H^k\|$ is uniformly bounded for all k -s, contrary to Lemma 4.3.

If $\|H'\| \geq 1$, then Lemma 4.10 gives a reduced computation $\pi_{21}(w_0) \rightarrow \dots \rightarrow \pi_{21}(w_t)$ with history H' . As above we can obtain reduced computations $\pi_{21}(w_0) \rightarrow \dots$ of the S -machine M_1 with histories H'^k . For $k \geq 3$, Lemma 4.14 (2) implies that all words in the computation with history H are positive. Then the same property must be true for the computation with history H' , and by Lemma 3.8, it is also a computation of the Turing machine M_1 with the same history H' , contrary to Lemma 2.3 (c). Thus the lemma is proved by contradiction. \square

Lemma 4.16. (a) *Let X_2 be the set of all words $\pi_{1,2}(W)$ accepted by M_2 , where W is an input word of the Turing machine M_1 . Then a word W' belongs to X_2 if and only if $W' \equiv \pi_{1,2}(W)$, and W is an input word of M_1 accepted by the Turing machine M_1 . Hence the set of words accepted by M_2 is not recursive.*

(b) *For every $W' \equiv \pi_{1,2}(W) \in X_2$ there exists only one reduced computation of M_2 accepting W' , the length of that computation is between the length T of the reduced*

computation of M_1 accepting W (this computation is unique by Lemma 2.3 (e)) and $\exp(O(T))$.

Proof. (a) Let $W' \equiv \pi_{1,2}(W)$, where W is an input word of the Turing machine M_1 . Suppose that W is accepted by the (symmetric) Turing machine M_1 . By part (b) of Lemma 2.3, the history H of the accepting computation consists of positive commands only. Then the computation of M_2 with history $\Pi_{1,2}(H, W)$ accepts W' by Remark 4.12.

Suppose that W' is accepted by M_2 , and H' is the history of an accepting computation C' . By Lemma 4.10, $\Pi_{2,1}(H')$ is a reduced history of an accepting computation of the S -machine M_1 starting with the input admissible word $\pi_{2,1}(W')$. Therefore by part (c) of Lemma 2.2, the first rule in H' is $\bar{\theta}$ for some positive rule θ of M_1 . Again by Lemma 2.2 (c), the last rule of $\Pi_{2,1}(H')$ is positive. It follows that the last rule of H' must be $\zeta(\theta')$ for some positive rule θ' since the accepted admissible word of M_2 has state letters having no θ -indices. By Lemma 4.14, then all words in computation C' are positive because both W and $W' \equiv \pi_{1,2}(W)$ are positive too. Therefore all words in the accepting computation $W \rightarrow \dots \rightarrow \pi_{2,1}(W' \cdot H')$ of the S -machine M_1 are positive. By Lemma 3.8, the latter computation is an accepting computation of the Turing machine M_1 , whence $W \in X_1$.

(b) The existence and the uniqueness follow from part (a) of this lemma and Lemmas 4.6 and 4.11. The part about the length of computation follows from Lemma 4.2 (2). \square

Similarly to the case of Turing machines, for every function $f(n)$, we define f -good numbers for M_2 . We call a number b f -good provided for every input word W from X_2 , if the length of the input sector (that is the Q_0P_1 -sector) of W is $< b$, then $f(T) \leq b$, where T is the time of accepting W by M_2 . Now Theorem 14.1 and Lemma 4.16 (b) imply

Lemma 4.17. *For every $\alpha > 0$, the set of $\exp(\alpha n)$ -good numbers of M_2 is infinite.*

Remark 4.18. The machine $Z(A)$ and the copies of it $Z^{(\theta,i)}$ do not satisfy Property 3.1 since two a -letters are involved in the (copies of) rules $r_{12}(a)$. Therefore further we will use the machine \tilde{M}_2 obtained from M_2 by the application of Lemma 3.3. Note that the claim of Lemma 4.15 is correct for \tilde{M}_2 as well. Indeed if the words $w_0 \equiv w_t$ in a computation $w_0 \rightarrow \dots \rightarrow w_t$ of \tilde{M}_2 with non-empty reduced history H involve auxiliary state letters, then there is a computation of \tilde{M}_2 with a reduced history H' , where H' is a freely conjugate of H , which starts and ends with the words $w'_0 \equiv w'_t$, having no special state letters. Then Lemma 3.3 and Lemma 4.15 for M_2 lead to a contradiction. Since the modification of M_2 does not touch the rules $\bar{\theta}$ corresponding to the rules θ of M_1 , the statements of Lemmas 4.16 and 4.17 also remain valid for \tilde{M}_2 . Thus Lemmas 4.15, 4.16, and 4.17 will be applied to the modified machine \tilde{M}_2 . Moreover, it follows from the definitions of M_2 and \tilde{M}_2 that these S -machines inherit the Property (c) from Lemma 2.2 of the Turing machine M_1 (for positive rules).

If the sum from Property 3.1 (2) is positive for some rule of \tilde{M}_2 , then this sum is 1, and we have $\|v_i\| + \|v'_i\| = 1$ or $\|u_{i+1}\| + \|u'_{i+1}\| = 1$ for a unique i . In the first case (in the second case) we say that the rule is *left* (is *right*).

Remark 4.19. Note that similarly, we can define the \circ -product $\mathcal{S} \circ \mathcal{S}'$ of any S -machine \mathcal{S} and S -machine $\mathcal{S}' = \mathcal{S}'(A)$ depending on the tape alphabet A . Furthermore, one can replace the auxiliary machine \mathcal{S}' by several S -machines $\mathcal{S}_1, \dots, \mathcal{S}_d$. Namely, one inserts a p -letter between two consecutive state letters in the standard base of \mathcal{S} and treats any

subword $q_i \dots p \dots q_{i+1}$ as an admissible subword for \mathcal{S}_i -s. For each rule θ of \mathcal{S} , one has a modified rule $\bar{\theta}$ of the composition. The application of the rule $\bar{\theta}$ is normally framed by alternated works of the auxiliary machines $\mathcal{S}_j^{(\theta, i)}$, and the priorities of the work of these machines may depend on θ . We are not going to define this construction formally, leaving it to the reader. In the next subsection, we shall introduce the \circ -product of \tilde{M}_2 and two primitive S -machines.

4.2 The machine M_3

Let M_2 be the S -machine $M_1 \circ Z$ and \tilde{M}_2 the modification from Remark 4.18. For every set of letters A , let A' and A'' be disjoint copies of A , the maps $a \mapsto a'$ and $a \mapsto a''$ identify A with A' and A'' , resp. Let $\vec{Z} = \vec{Z}(A)$ and $\overleftarrow{Z} = \overleftarrow{Z}(A)$ be the S -machines with tape alphabet $A' \sqcup A''$, state alphabet $\{L\} \cup P \cup \{R\}$, where $P = \{p(1), p(2), p(3)\}$ and the following positive S -rules. For \vec{Z} we have the rules

$$\xi_1(a) = [L \rightarrow L, p(1) \rightarrow a'p(1)(a'')^{-1}, R \rightarrow R], a \in A$$

Comment: The head moves from left to right, replacing the word on the tape by its copy in the alphabet A' .

$$\xi_2 := [L \rightarrow L, p(1) \xrightarrow{\ell} p(2), R \rightarrow R]$$

Comment: When the head meets R , it turns into $p(2)$.

For \overleftarrow{Z} , we define the rules

$$\xi_3(a) = [L \rightarrow L, p(2) \rightarrow (a')^{-1}p(2)a'', R \rightarrow R]$$

Comment: The head $p(2)$ moves from right to left, replacing the word in A' by its copy in A'' .

$$\xi_4 = [L \xrightarrow{\ell} L, p(2) \rightarrow p(3), R \rightarrow R]$$

Comment: When the head reaches the left end of the tape, it turns into $p(3)$.

Remark 4.20. For every $a \in A$, $i = 1, 3$, it will be convenient to denote $\xi_i(a)^{-1}$ by $\xi_i(a^{-1})$. It is clear from the definition $\xi_i(a)$ that this does not lead to a confusion.

Remark 4.21. Note that if the machine \vec{Z} (the machine \overleftarrow{Z}) starts with the word $Lp(1)u''R$ (resp., with $Lu'p(2)R$) and ends with the word $Lu'p(2)R$ (with $Lp(3)u''R$), where u'' is the word in A'' -letters, then the history H of the only reduced computation such that $Lp(1)u''R \cdot H = Lu'p(2)R$ (such that $Lu'p(2)R \cdot H = Lp(3)u''R$) is

$\xi_1(a_1) \dots \xi_1(a_m) \xi_2$ (is $\xi_3(a_m) \dots \xi_3(a_1) \xi_4$) and its length is $\|u\| + 1$. Here $u \equiv a_1 \dots a_m$ is the copy of u' (of u'') in the alphabet A .

Similarly, any reduced computation of \vec{Z} (of \overleftarrow{Z}) ending with $Lu'p(2)R$ (resp., with $Lp(3)u''R$) is uniquely determined by its initial admissible word and has length $\leq \|u\| + 1$.

Remark 4.21 implies, in particular

Lemma 4.22. Suppose that $W \rightarrow \dots \rightarrow W \cdot H$ is a reduced computation of \vec{Z} (of \overleftarrow{Z}) with history H and the standard base. Suppose that both W and $W \cdot H$ contain $Lp(1)$ or both contain $p(2)R$ (resp., $p(2)R$ or $Lp(3)$). Then H is empty.

Below we define M_3 as $\tilde{M}_2 \circ \{\overrightarrow{Z}, \overleftarrow{Z}\}$ (see Remark 4.19), that is we insert copies of \overleftarrow{Z} and \overrightarrow{Z} between every two consecutive state letters of \tilde{M}_2 . We simplify and unify the notation by changing the value of N and renaming the parts of the state alphabet of \tilde{M}_2 . In this section we assume that \tilde{M}_2 has the standard base $s_0 s_1 \dots s_N$ (and forget more detailed earlier notations).

For every $i = 1, \dots, N$, we make copies Y'_i and Y''_i of the alphabet Y_i of \tilde{M}_2 ($i = 1, \dots, N$). Let Θ be the set of positive commands of \tilde{M}_2 . The set of state letters of M_3 is

$$S_0 \cup P_1 \cup S_1 \cup P_2 \cup \dots \cup P_N \cup S_N$$

where $P_i = \{p^{(i)}, p^{(i,1)}, p^{(i,0)}, p^{(\theta,i)}(1), p^{(\theta,i)}(2), p^{(\theta,i)}(3) \mid \theta \in \Theta\}$ $i = 1, \dots, N$, $S_i = Q_i \sqcup Q_i \times \Theta$ where $Q_0 \sqcup Q_1 \sqcup \dots \sqcup Q_N$ is the set of state letters of \tilde{M}_2 . Thus the state letters L and R of the copies of machines \overrightarrow{Z} and \overleftarrow{Z} are identified with the corresponding S -letters as in the case of $M_2 = M_1 \circ Z$. We shall call the state letters from P_i -s the *control state letters* or *p-letters*, and the other state letters (i.e. the copies of the state letters of \tilde{M}_2), the *basic state letters*.

The set of tape letters of M_3 is $Y = Y_1 \sqcup \dots \sqcup Y_{2N} = Y'_1 \sqcup Y''_1 \sqcup Y'_2 \sqcup Y''_2 \sqcup \dots \sqcup Y'_N \sqcup Y''_N$.

Let θ be a positive \tilde{M}_2 -rule which is not a right rule. Assume θ is of the form

$$[s_0 \rightarrow s'_0, v_1 s_1 \rightarrow v'_1 s'_1, \dots, v_N s_N \rightarrow v'_N s'_N],$$

where $s_i, s'_i \in S_i$, and v_i -s are words in Y . Then this rule is replaced in M_3 by positive

$$\bar{\theta} = \left[\begin{array}{l} s^{(\theta,0)} \xrightarrow{\ell} (s')^{(\theta,0)}, p^{(\theta,1)}(1) \rightarrow p^{(\theta,1)}(1), v_1 s^{(\theta,1)} \xrightarrow{\ell} v'_1 (s')^{(\theta,1)}, \\ p^{(\theta,2)}(1) \rightarrow p^{(\theta,2)}(1), \dots, v_N s^{(\theta,N)} \rightarrow v'_N (s')^{(\theta,N)} \end{array} \right]$$

with $Y_{2i-1}(\bar{\theta}) = \emptyset$ and $Y_{2i}(\bar{\theta}) = Y''_i(\theta)$. As an exception, the left-hand sides of the parts of $\bar{\theta}$ are of the form $s_0 \xrightarrow{\ell}, p^{(1,1)} \rightarrow, s_1 \xrightarrow{\ell}, p^{(2,1)} \rightarrow, \dots, s_N \rightarrow$ if θ is the unique start rule, i.e., they do not depend on the index θ .

A right positive rule θ of the form

$$[s_0 u_1 \rightarrow s'_0 u'_1, s_1 u_2 \rightarrow s'_1 u'_2, \dots, s_N \rightarrow s'_N]$$

is replaced by the right rule $\bar{\theta}$ of M_3 :

$$\bar{\theta} = \left[\begin{array}{l} s^{(\theta,0)} u_1 \rightarrow (s')^{(\theta,0)} u'_1, p^{(\theta,1)}(2) \xrightarrow{\ell} p^{(\theta,1)}(2), s^{(\theta,1)} u_2 \rightarrow (s')^{(\theta,1)} u'_2, \\ p^{(\theta,2)}(2) \xrightarrow{\ell} p^{(\theta,2)}(2), \dots, s^{(\theta,N)} \rightarrow (s')^{(\theta,N)} \end{array} \right]$$

with $Y_{2i-1}(\bar{\theta}) = Y'_i(\theta)$ and $Y_{2i}(\bar{\theta}) = \emptyset$.

Now we want to describe the alternating work of the auxiliary machines $\overleftarrow{Z}^{(\theta,i)}$ and $\overrightarrow{Z}^{(\theta,i)}$. Normally each of them is switched on exactly once in the frame of the rule θ , but the sequence of their turning on depends on θ .

First, we need the following *transition* rule $\zeta_-(\theta)$. This rule adds θ to all state letters and turns all $p^{(j)}$ into $p^{(\theta,j)}(1)$:

$$[s_i \xrightarrow{\ell} s^{(\theta,i)}, p^{(j)} \rightarrow p^{(\theta,j)}(1), i = 0, \dots, N, j = 1, \dots, N]$$

so that the rule $\bar{\theta}$ becomes applicable if θ is not a right rule. Again, as an exception, we do not introduce $\zeta_-(\theta)$ for the start rule $\theta = \theta_{start}$ of \tilde{M}_2 .

If θ is a right rule, then the rule $\zeta_-(\theta)$ successively switches on the machines $\overrightarrow{Z}^{(\theta,1)}, \dots, \overrightarrow{Z}^{(\theta,N)}$. (We will not present formulas for the rules $\bar{\tau}_i(\theta)$ as in the definition of M_2 , since the explicit form of these rules are not necessary.) Then the state letters $p^{(\theta,j)}(1)$ ($j = 1, \dots, N$) successively turn into $p^{(\theta,j)}(2)$, find themselves just before s_i -letters, and the rule $\bar{\theta}$ can be applicable.

After an application of a non-right rule $\bar{\theta}$, the machines $\overrightarrow{Z}^{(\theta,j)}$ move the p -letters to the write, change $p^{(\theta,j)}(1)$ by $p^{(\theta,j)}(2)$, and then the machines $\overleftarrow{Z}^{(\theta,j)}$ move the p -letters to the left and change $p^{(\theta,j)}(2)$ by $p^{(\theta,j)}(3)$. After an application of a right rule $\bar{\theta}$, only machines $\overleftarrow{Z}^{(\theta,j)}$ work.

Finally, the transition rule $\zeta_+(\theta)$ removes index θ from all state letters, and turns all $p^{(\theta,j)}(3)$ into $p^{(j)}$:

$$[s^{(\theta,i)} \xrightarrow{\ell} s_i, p^{(\theta,j)}(3) \rightarrow p^{(j)}, i = 0, \dots, N, j = 1, \dots, N].$$

If θ is the unique accept rule of \tilde{M}_2 then all $p^{(j)}$ -s in the above definition of $\zeta_+(\theta)$ must be replaced by special letters $p^{(j,0)}$ -s.

An important specification is the following. If θ is a right rule and $\|u_{i+1}\| + \|u'_{i+1}\| = 1$ (see (**)), then the application of $\bar{\theta}$ always switches on the auxiliary machines in the order $\overleftarrow{Z}^{(\theta,i)}, \overleftarrow{Z}^{(\theta,i-1)}, \dots, \overleftarrow{Z}^{(\theta,1)}, \overleftarrow{Z}^{(\theta,N)}, \dots, \overleftarrow{Z}^{(\theta,i+1)}$. If θ is a left rule and $\|v_i\| + \|v'_i\| = 1$, then $\bar{\theta}$ must successively start up $\overrightarrow{Z}^{(\theta,i+1)}, \dots, \overrightarrow{Z}^{(\theta,N)}, \overrightarrow{Z}^{(\theta,1)}, \overrightarrow{Z}^{(\theta,i)}$. If θ is neither right nor left, then the order for the first N machines is $\overrightarrow{Z}^{(\theta,1)}, \dots, \overrightarrow{Z}^{(\theta,N)}$.

For every admissible word w of \tilde{M}_2 with standard base, let $\pi_{2,3}(w)$ be the admissible word of M_3 obtained by inserting control state letters $p^{(i)}$ ($p^{(i,1)}$ or $p^{(i,0)}$ if the word w has state letters from the start vector \vec{s}_1 , resp., from the accept vector \vec{s}_0 of \tilde{M}_2) next to the right of each s_{i-1} , $i \leq N$. The stop word of M_3 is $\pi_{2,3}(\pi_{1,2}(W_0))$, where W_0 is the stop word of M_1 . For every input word w of \tilde{M}_2 we call $\pi_{2,3}(w)$ an *input* word of M_3 .

Remark 4.23. It follows from Remark 4.18 and the definition of M_3 that the S-machine M_3 inherits the Property (c) from Lemma 2.2 (for positive rules).

The $p^{(1)}s_1$ -sector of an admissible word of M_3 is called the *input sector* of that word.

Assume that $w \rightarrow w \cdot \theta$ is a computation of the machine \tilde{M}_2 with standard base and a positive rule θ . Then, by the definition of M_3 , we have the canonically defined reduced computation $\dots \rightarrow \pi_{2,3}(w) \rightarrow \pi_{2,3}(w) \cdot \bar{\theta} \rightarrow \dots$ starting and ending with words whose state letters have no θ -indices and all other words do have θ -indices. The computation of M_3 with these properties is unique since the base is standard. Indeed Remark 4.21 and the definition of M_3 uniquely determine the order of rules for each of the auxiliary machines $\overrightarrow{Z}^{(\theta,j)}$ and $\overleftarrow{Z}^{(\theta,j)}$. (For example, a machine $\overrightarrow{Z}^{(\theta,j)}$ can start working only if the state letter $p^{(\theta,j)}(1)$ is the right neighbor of a letter $s^{(\theta,j-1)}$ since the $p^{(\theta,j)}(1)s^{(\theta,j-1)}$ -sector is locked before the start, and $\overrightarrow{Z}^{(\theta,j)}$ cannot finish its work until the p -letter becomes the left neighbor of $s^{(\theta,j)}$ and turns into $p^{(\theta,j)}(2)$, etc.) Thus the following claim is true.

Lemma 4.24. *For every computation $w \rightarrow w \cdot \theta$ of the machine \tilde{M}_2 with standard base and a positive rule θ , there is a unique reduced M_3 -computation $\dots \rightarrow \pi_{2,3}(w) \rightarrow \pi_{2,3}(w) \cdot \bar{\theta} \rightarrow \dots$ such that it starts and ends with words whose state letters have no θ -indices and all other words have θ -indices. The history of this computation starts with $\zeta_-(\theta)$ and ends with $\zeta_+(\theta)$.*

We denote the history of this computation by $\Pi_{2,3}(\theta, w)$. It follows from Remark 4.21 that the length of this history is $1 + 2(|w|_a + N)$, where $|w|_a$ is the number of a -letters in the word w . If θ is a negative rule of \tilde{M}_2 , then we invert the computation constructed for θ^{-1} , and so $\Pi_{2,3}(\theta, w) \equiv \Pi_{2,3}(\theta^{-1}, w \cdot \theta)^{-1}$.

Similarly, with arbitrary reduced computation $w \rightarrow \dots \rightarrow w \cdot H$ with the standard base of \tilde{M}_2 and having a history $H \equiv \theta_1 \dots \theta_t$ we associate the reduced computation of M_3 with history

$$\Pi_{2,3}(H, w) \equiv \Pi_{2,3}(\theta_1, w) \Pi_{2,3}(\theta_2, w \cdot \theta_1) \dots \Pi_{2,3}(\theta_t, w \cdot \theta_1 \dots \theta_{t-1})$$

It follows from the previous paragraph that $\|H\| \leq \|\Pi_{2,3}(H, w)\| = O(\|H\|^2)$ for every accepted computation. Indeed $|w|_a = O(\|H\|)$ since the stop word has no tape letters.

Recall that only rules of the form $\zeta_{\pm}(\theta)$ and the start rule involve state letters without θ -indices. Therefore it follows from Lemma 4.24 that every reduced computation with the standard base of M_3 starting and ending with the admissible words without θ -indices in their state letters, has history of the form $\Pi_{2,3}(H, w)$ for some reduced computation $w \rightarrow \dots \rightarrow w \cdot H$ of \tilde{M}_2 . In particular, our discussion and Lemma 4.16 (b) imply the following

Lemma 4.25. (a) *Let X_3 be the set of all words $\pi_{2,3}(w)$ accepted by M_3 , where w is an input word of \tilde{M}_2 (or M_2). Then a word W belongs to X_3 if and only if $W = \pi_{2,3}(w)$, and $w \in X_2$. Hence the set of words accepted by M_3 is not recursive.*

(b) *For every $W \equiv \pi_{2,3}(w) \in X_3$ there exists only one reduced computation of M_3 accepting W , the length of that computation is between the length T of the reduced computation of \tilde{M}_2 accepting w and $O(T^2)$.*

We can define f -good numbers of M_3 in a similar way as for M_2 . Lemmas 4.17 and 4.25 imply

Lemma 4.26. *For every constant $c > 0$, the set of $\exp(cn)$ -good numbers of M_3 is infinite.*

As in the previous section, we need to define more maps between S -machines \tilde{M}_2 and M_3 .

For every admissible word W of M_3 with the standard base, let $\pi_{3,2}(W)$ be the word obtained by removing state p -letters, θ -indices of state letters, and the indices that distinguishes a -letters from the left and from the right of p -letters. We obtain an admissible word of \tilde{M}_2 . Note that we have

$$\pi_{3,2}(\pi_{2,3}(w)) \equiv w. \quad (4.4)$$

For every rule $\bar{\theta}$ of M_3 corresponding to a rule θ of \tilde{M}_2 we denote $\theta = \Pi_{3,2}(\bar{\theta})$. If $\bar{\theta}$ does not correspond to a rule of \tilde{M}_2 , we denote by $\Pi_{3,2}(\bar{\theta})$ the empty rule. The map $\Pi_{3,2}$ extends to histories of computations in the natural way.

Remark 4.27. It can be proved similarly to Lemma 4.10, that if H is a history of a computation of M_3 with standard base and $W \cdot H = W'$, then $\Pi_{3,2}(H)$ is reduced and

$$\pi_{3,2}(W) \cdot \Pi_{3,2}(H) \equiv \pi_{3,2}(W'). \quad (4.5)$$

Lemma 4.28. *Suppose a commutation $W \rightarrow \dots$ of M_3 with a base B has a reduced history $H \equiv \dots \bar{\theta}_1 H' \bar{\theta}_2^\eta \dots$, where $\Pi_{3,2}(H) \equiv \theta_1 \theta_2^\eta$ for some positive θ_1 and θ_2 , and $\eta = \pm 1$.*

- (1) *if B is standard, then the word $W \cdot \bar{\theta}_1$ is completely determined by H' ;*
- (2) *if $\theta_2^\eta \neq \theta_1^{-1}$, then B or B^{-1} is a subword of the standard base of the machine M_3 ;*
- (3) *let $\|u_{j+1}\| + \|u'_{j+1}\| = 1$ ($\|v_j\| + \|v'_j\| = 1$) for the rule θ_1 , and B is not a subword of the standard base or of its inverse. Then $\theta_2^\eta \equiv \theta_1^{-1}$, and no rule from H' locks the $s^{(\theta,j)}p^{(\theta,j+1)}$ -sector (resp., the $p^{(\theta,j)}s^{(\theta,j)}$ -sector).*

Proof. (1) The argument used for Lemma 4.24 shows that since the base is standard, each of the machines $\bar{Z}^{(\theta,j)}$ must accomplish its standard work after the application of the rule $\bar{\theta}_1$. Therefore the history of the work of $\bar{Z}^{(\theta,j)}$ completely determines the $p^{(j)}s_j$ -sector subword of the word $W \cdot \bar{\theta}_1$. The θ -indices of the state letters of this word are obviously determined by the histories of $\bar{Z}^{(\theta,j)}$, and Statement (1) is proved.

(2) The assumptions implies that the θ -indices that the state letters have after the application of $\bar{\theta}_1$, must disappear earlier than one applies $\bar{\theta}_2^\eta$. Again by Remark 4.21, it follows that each of the machines $\bar{Z}^{(\theta_1,j)}$ must perform its standard work. Therefore for every $j = 1, \dots, N$, the history H' has rules locking $s^{(\theta,j-1)}p^{(\theta,j)}(1)$ -sectors and it has rules locking $p^{(\theta,j)}(2)s^{(\theta,j)}$ -sectors. Hence, by Lemma 3.4, the base B has no subwords of the form $q^{\pm 1}q^{\mp 1}$, and so $B^{\pm 1}$ is a subword of the standard base by the definition of admissible word.

(3) First of all, we have $\theta_2^\eta \equiv \theta_1^{-1}$ by Property (2). Then we assume that a right rule τ from H' locks the $s^{(\theta,j)}p^{(\theta,j+1)}$ -sector.

The locking rule τ must belong to the machine $\bar{Z}^{(\theta_1,j+1)}$ since other auxiliary machines working after the application of the right rule $\bar{\theta}_1$ do not lock this sector. Taking into account the order of the work of auxiliary machines after an application of a right rule, we conclude that the machines $\bar{Z}^{(\theta_1,j)}, \dots, \bar{Z}^{(\theta_1,1)}, \bar{Z}^{(\theta_1,N)}, \dots, \bar{Z}^{(\theta_1,j+2)}$ works before the machine $\bar{Z}^{(\theta_1,j+1)}$ starts working. Since the last rule of $\bar{Z}^{(\theta_1,j+2)}$ locks the $p^{(\theta,j+1)}s^{(\theta,j+1)}$ -sector, H' has a rule locking $p^{(\theta,j+1)}s^{(\theta,j+1)}$ -sector. Proceeding in this manner, we then consider the work of the preceding machine $\bar{Z}^{(\theta_1,j+2)}$ and conclude that the $s^{(\theta,j+1)}p^{(\theta,j+2)}$ -sectors was locked by by some rules from H' . Finally, we see that every sector except for $p^{(\theta,j)}s^{(\theta,j)}$ was locked by some rule from H' . The $p^{(\theta,j)}s^{(\theta,j)}$ -sector was locked by $\bar{\theta}_1$ since θ_1 is a right rule. By Lemma 3.4, B is a subword of the standard base or of its inverse, a contradiction.

Similar argument works if θ_1 is a left rule. In this case if $p^{(\theta,j)}s^{(\theta,j)}$ -sector is locked by a rule from H' , then $\bar{\theta}_1$ switches on the machines $\bar{Z}^{(\theta_1,j+1)}, \dots, \bar{Z}^{(\theta_1,N)}, \bar{Z}^{(\theta_1,1)}, \dots, \bar{Z}^{(\theta_1,j)}$, and we again come to a contradiction. □

Lemma 4.29. *Suppose that the admissible word W of M_3 has the standard base. Suppose that a reduced computation applicable to W has history of the form H^3 . Then H does not contain rules $\bar{\theta}$ corresponding to the rules θ of the S -machine \tilde{M}_2 .*

Proof. Suppose that H contains a rule $\bar{\theta}_0^{\pm 1}$. Then $\bar{\theta}_0$ occurs in a subword H' of the form

$$H' \equiv (\bar{\theta}_0^{\epsilon_0} H_1 \bar{\theta}_1^{\epsilon_1} H_2 \bar{\theta}_2^{\epsilon_2} \dots H_s)^2 \bar{\theta}_0^{\epsilon_0}$$

where all θ_i are positive rules from \tilde{M}_2 , $\epsilon_i \in \{-1, 1\}$, and H_i consists of rules of various copies of the S -machines \overleftarrow{Z} and \overrightarrow{Z} . We can assume that $\epsilon_0 = 1$ (if not, we can replace H by H^{-1}).

Let W be the initial word of the computation with history H_1 . Then the word $W \cdot \bar{\theta}_0$ is completely determined by H_1 by Lemma 4.28. Similarly, the word

$$W'' \equiv W \cdot (\bar{\theta}_0 H_1 \bar{\theta}_1^{\epsilon_1} H_2 \bar{\theta}_2^{\epsilon_2} \dots H_s) \bar{\theta}_0$$

is determined by the same H_1 . Thus $W' \equiv W''$, but then by Remark 4.27, the equal words $\pi_{32}(W')$ and $\pi_{32}(W'')$ are connected by a non-empty reduced computation of the machine \tilde{M}_2 . This contradicts Lemma 4.15, and the lemma is proved. \square

4.3 The machine M_4

Recall that the set of state letters of M_3 is $S_0 \sqcup P_1 \sqcup S_1 \sqcup P_2 \dots \sqcup S_N$ where $\cup P_i$ contains the control state letters. Let $\Theta(3)$ be the set of positive rules of M_3 except for the start and the accept rules. We introduce two copies $Y(3)$ and $Y'(3)$ of $\Theta(3)$ which will be parts of the tape alphabet of M_4 . Let maps $\theta \mapsto y_\theta$ and $\theta \mapsto y'_\theta$ identify $\Theta(3)$ with $Y(3)$ and $Y'(3)$.

The standard base of M_4 is $tk s_0 p_1 s_1 \dots p_N s_N k' t'$, where $s_0 p_1 s_1 \dots p_N s_N$ is the standard base of M_3 . (Every state letter is called a q -letter as earlier, but from now on, we also can use t -, k -, s -, p -letters or s_0 -, p_1 -letters, and so on.) As for M_3 , the $p_1 s_1$ -sector of an admissible word is called the *input sector* of that word.

The new parts of the set of state letters are $T = \{t\}$, $K = \{k(1), k(2), k(3)\}$, $K' = \{k'(1), k'(2), k'(3)\}$, $T' = \{t'\}$.

The new sets of state letters are now denoted by $T, K, S_0, P_1, \dots, P_N, S_N, K', T'$. The set of tape letters in the tk -sector is $Y(3)$, the sets of tape letters in ks_0 -sector and in the $s_N k'$ -sector are empty, and the set of tape letters in the $k' t'$ -sector is $Y'(3)$, $k \in \{k(1), k(2), k(3)\}$, $k' \in \{k'(1), k'(2), k'(3)\}$. The tape letters in the other sectors are as in M_3 .

The positive rules of the machine M_4 are divided into three *Steps*. Each rule below contains subrules $s_0 \rightarrow s_0$, $t \rightarrow t$ and $t' \rightarrow t'$, so we sometimes omit these subrules.

Step 1.

$$\rho_1(y) = \left[\begin{array}{l} k(1) \xrightarrow{\ell} yk(1), p^{(1,1)} \rightarrow p^{(1,1)}, s_i \xrightarrow{\ell} s_i (1 \leq i \leq N), \\ p^{(1,j)} \xrightarrow{\ell} p^{(1,j)} (2 \leq j \leq N), k'(1) \xrightarrow{\ell} k'(1) \end{array} \right], y \in Y(3)$$

where the s - and p -letters form the start vector \vec{s}_1 for the machine M_3 .

Comment: The machine writes the $Y(3)$ -copy of a (positive) history word in the tk -sector to the left of k_1 . The word between k and k' is an input word of M_3 . All sectors except for the tk -sector and the $p^{(1,1)} s_1$ -sector are locked by the rules of Step 1.

Transition rule (12) from Step 1 to Step 2 is the ‘extension’ $\theta(M_4)$ of the unique start rule $\theta = \theta_{start}$ of the machine M_3 :

$$(12) = \left[k(1) \xrightarrow{\ell} k(2), \dots, s_N \xrightarrow{\ell} s'_N, k'(1) \xrightarrow{\ell} k'(2) \right],$$

where the parts of the rule (12) between k - and k' -letters are the parts of θ_{start} .

Comment: After that rule is applied, the machine is ready to execute copies of the machines $\overrightarrow{Z}^{(\theta_{start,j})}$ and $\overleftarrow{Z}^{(\theta_{start,j})}$, on tapes 1 through N . All sectors except the tk -sector and the $p^{(1,1)} s_1$ -sector are locked by this rule.

Step 2. For every $\theta \in \Theta(3)$:

$$\theta(M_4) = [k(2) \xrightarrow{\ell} y_\theta^{-1}k(2), \theta, k'(2) \rightarrow k'(2)y'_\theta]$$

Comment: On tapes 1 through N , the machine executes (backwards) the history written in the tk -sector, erases the word in that sector, and copies it to the $k't'$ -sector.

Transition rule (23) $= \theta(M_4)$ from Step 2 to Step 3 ‘extends’ the accept rule $\theta = \theta_{\text{accept}}$ of M_3 :

$$(23) = [t \xrightarrow{\ell} t, k(2) \xrightarrow{\ell} k(3), \dots, s_N \xrightarrow{\ell} s'_N, k'(2) \rightarrow k'(3), t \rightarrow t']$$

where the parts of the rule (23) between k - and k' -letters are the parts of θ_{accept} .

Comment. All sectors except for the $k't'$ -sector are locked by this rule.

Step 3.

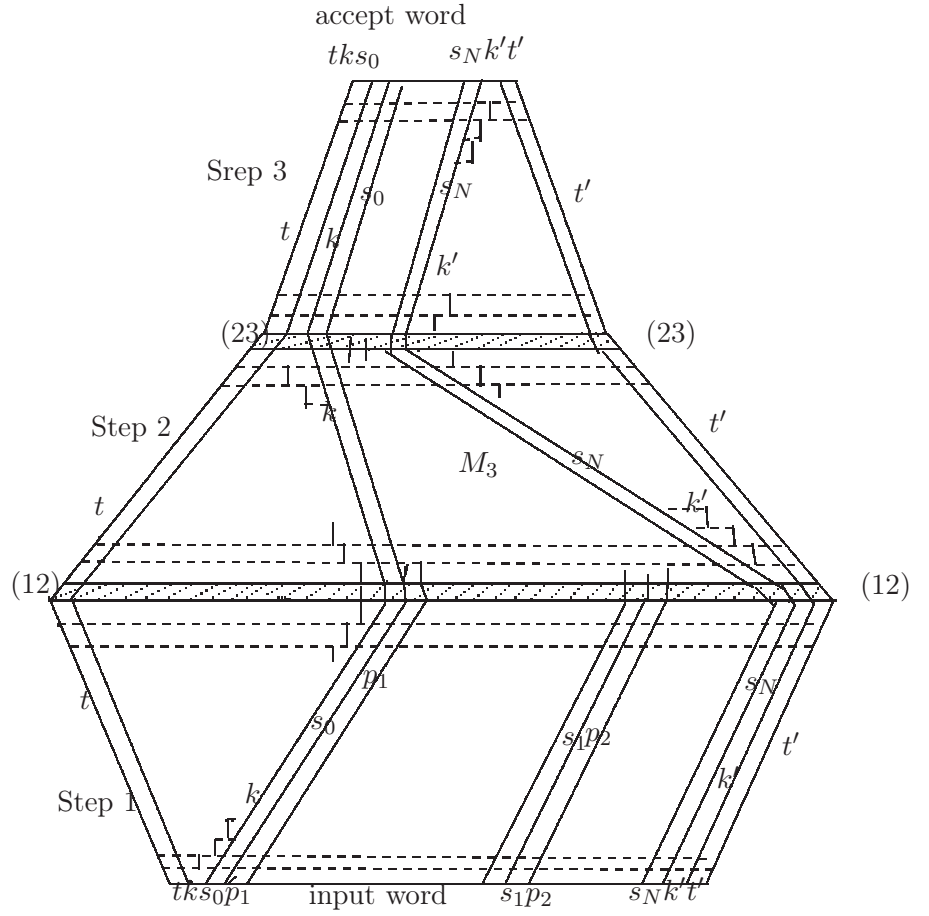
$$\rho_3(\theta) = [t \xrightarrow{\ell} t, k(3) \xrightarrow{\ell} k(3), \dots, s'_N \xrightarrow{\ell} s'_N, k'(3) \rightarrow k'(3)(y'_\theta)^{-1}, t' \rightarrow t'],$$

where the state letters between k - and k' -letters form the accept vector \vec{s}'_0 of M_3

Comment: The machine erases the history from the $k't'$ -sector. All other sectors are locked by the rules of this Step.

For every admissible input word $W \in X_3$ of M_3 let $\pi_{3,4}(W) \in X_4$ be the admissible word of M_4 obtained by adding state letters $k(1), k'(1), t, t'$, hence $\pi_{3,4}(W) \equiv k(1)tWt'k'(1)$. For every input word W of M_3 we call the word $\pi_{3,4}(W)$ an *input* word of M_4 . The stop word of M_4 , W_{M_4} , is obtained from the stop word W_{M_3} of M_3 by adding state letters $k(3), k'(3), t, t'$, i.e., $W_{M_4} \equiv k(3)tW_{M_3}t'k'(3)$.

Remark 4.30. From now on, we do not show the indices (i) ($i = 1, 2, 3$) of the letters k , and k' assuming that the indices are appropriate for an admissible word.



Normal work of M_4 from a start word to the accept word

For every accepting computation $W \rightarrow W \cdot \theta_1 \rightarrow W \cdot \theta_1 \theta_2 \rightarrow \dots \rightarrow W \cdot \theta_1 \dots \theta_n \equiv W_{M_3}$ (where θ_i are rules and W is an input word for M_3) of M_3 with history $H \equiv \theta_1 \theta_2 \dots \theta_n$, $W \in X_3$, one canonically constructs a computation of M_4 : $\pi_{3,4}(W) \rightarrow \dots \rightarrow W_{M_4}$. The history of that computation is denoted by $\Pi_{3,4}(H)$. That computation first uses rules of Step 1 and writes a mirror copy of H' (i.e. H without the start and the accept rules) in the alphabet $Y(3)$ in the tk -sector, then executes rule (12), then executes the computation with history H' on the subword between k and k' while erasing the word in the tk -sector and moving it onto the $k't'$ -sector (written in $Y'(3)$). After H' is completed, rule (23) is executed, then the $k't'$ -sector is erased using rules of Step 3. Let $X_4 = \pi_{3,4}(X_3)$. Every word from this set of input configurations is accepted by M_4 . To simplify the notation, we can include the rules (12) and (23) to Step 2.

Suppose that a history of computation of M_4 has the form $H \equiv H_1 H_2 \dots$, where all rules of each H_i belong to the same Step j_i , and H_i is a maximal subword of H with this property. Then we say that the *step history* of that computation is $(j_1)(j_2) \dots$ (or that H is of type $(j_1)(j_2) \dots$). The following lemma is a straightforward consequence of the definition of M_4 and will be used without reference throughout the paper.

Lemma 4.31. *Every 2-letter subword of any step history of a computation of M_4 (with any base) is one of the following words: $(1)(2), (2)(1), (2)(3), (3)(2)$. Two consecutive steps are separated by $(12)^{\pm 1}$ or by $(23)^{\pm 1}$, resp., and the letters of the history neighboring any (12) (or (23)) from the left and from the right belong to different Steps.*

Lemma 4.32. *An admissible word of M_4 is not in the domain of the reduced histories of types:*

- (a) $(1)(2)(1)$ if the base of W has subword $k't'$;
- (b) $(3)(2)(3)$ if the base of W has subword tk ;
- (c) $(3)(2)(1)(2)(3)$ if the base of W is standard.

Proof. Cases (a) and (b) are almost identical, so suppose that the history H contains a subword $(12)H(12)^{-1}$, where H is of type (2). The word $V \equiv W \cdot (12)$ from the computation that is in the domain of H must have the subword between k' and t' empty (since it is in the domain of $(12)^{-1}$). Similarly, the word $V \cdot H$ must have the subword between k' and t' empty. If $H \equiv \theta_1(M_4)^{\pm 1} \dots \theta_s(M_4)^{\pm 1}$ where θ_i are positive rules of M_3 , then the subword between k' and t' in $V \cdot H$ is equal to $(y'_{\theta_s})^{\pm 1} \dots (y'_{\theta_1})^{\pm 1}$. Since this word is empty, we conclude that H is not reduced, a contradiction.

Suppose that the step history is of the form (c). Then the history H has the form $H_3(23)^{-1}H_2(12)^{-1}H_1(12)H'_2(23)H'_3$, where H_i, H'_i contain rules from Step i only. Restricting the computation to the subwords between k and k' of the admissible words, we obtain two reduced accepting computations of M_3 with the same initial word from X_3 and histories H_2^{-1}, H'_2 (this follows from the definitions of the rules of Step 2). By Lemma 4.25 (b) $H_2^{-1} \equiv H'_2$. Since every rule of Step 1 multiplies the tk -sector of the admissible word by an a -letter uniquely determined by the rule, the tk -sectors A_{tk}, B_{tk} in the words $W \cdot H_3(23)^{-1}H_2$ and $W \cdot H_3(23)^{-1}H_2(12)^{-1}H_1(12)$ respectively are the same. Since every rule of Step 1 multiplies that sector by a letter uniquely determined by that rule, we deduce that a copy of the word H_1 multiplied by A_{tk} is A_{tk} . Hence H_1 is empty, which contradicts the assumption that the computation is reduced. \square

Lemma 4.33. *Suppose that $W \equiv tW_1kW_2k'W_3t'$ is an admissible word of M_4 with the standard base. Suppose that W is in the domain of a reduced history of the form $(12)H(23)$. Then*

- (1) H contains only rules from Step 2, $(12)H(23) \equiv \theta_1(M_4) \dots \theta_n(M_4)$ for some rules $\theta_1, \dots, \theta_n$ of M_3 .
- (2) The word W_2 is from X_3 and $\theta_1 \dots \theta_n$ is a computation of M_3 accepting W_2 .

Proof. Suppose that H contains rules from Step 1 or 3. Then it contains a subword of one of two forms $(23)^{-1}H_1(23)$ or $(12)H_1(12)^{-1}$ with H_1 consisting of rules of Step 2 which contradicts Lemma 4.32. This implies part (1) of the lemma.

Since W is in the domain of (12) , the subword W_2 an admissible input word of M_3 . Since $W \cdot (12)H(23)$ is in the domain of $(23)^{-1}$, the subword between k and k' is the stop word W_{M_3} of M_3 . This implies part (2) of the lemma. \square

Lemma 4.34. *Suppose that W is an admissible word of M_4 with the standard base. Then*

- (a) *The step history of any reduced computation starting with W is a subword of $(2)(1)(2)(3)(2)(1)(2)$.*
- (b) *The step history of any accepting reduced computations starting with W is a suffix of the word $(2)(1)(2)(3)$.*

Proof. Indeed, in every step history $(i_1)(i_2)\dots(i_s)$ of a reduced computation of M_4 , after (1) we should have (2), after (2) we should have (1) or (3), after (3) we should have (2). The statement then follows immediately from Lemma 4.32. \square

Lemma 4.35. *Suppose that a history H of a reduced computation of M_4 with standard base contains both $(12)^{\pm 1}$ and $(23)^{\pm 1}$.*

- (a) *The number of occurrences of $(12)^{\pm 1}$ or $(23)^{\pm 1}$ in H is at most 6.*
- (b) *Suppose that the computation is accepting. Then the number of occurrences of $(12)^{\pm 1}$ or $(23)^{\pm 1}$ in H is at most 3.*

Proof. Immediately follows from Lemmas 4.34 and 4.31. \square

Lemma 4.36. *Recall that X_4 is the set of all words of the form $\pi_{3,4}(W)$, $W \in X_3$.*

An input word $W' \equiv \pi_{3,4}(W)$ is accepted by M_4 if and only if $W \in X_3$. Hence the language accepted by M_4 is not recursive.

Proof. If $W \in X_3$ then $\pi_{3,4}(W) \in X_4$ since the corresponding computation was constructed together with the definition of M_4 . Let $W' \equiv \pi_{3,4}(W)$ for some admissible input word W of M_3 and H be the history of an accepting computation for W' . By Lemma 4.34 (b), $H \equiv H_1(12)H_2(23)H_3$, where H_j contains only rules of Step j ($j = 1, 2, 3$). By Lemma 4.33, W is in X_3 , and H_2 corresponds to a computation of M_3 accepting W . \square

Definition 4.37. Let $T_1 < T_2 < \dots$ be all the times of acceptance of acceptable input words of M_3 .

We will call a computation of M_4 *standard* if it has the standard base and history of the form $(12)(2)(23)$. The following lemma gives (almost) linear upper bounds for the lengths of many computations with standard base.

Lemma 4.38. (a) *Suppose that an admissible word W of M_4 is accepted by M_4 . Suppose that the length of a reduced accepting computation of W is not in $\cup_{i=1}^{\infty} (T_i, 9T_i)$. Then the length of this accepting computation of W is at most $6|W|_a$.*

(b) *Let b be an integer such that any standard computation starting with a word W with $|W|_a \leq b$ has the history of length $< \log b$. Suppose W is any accepted admissible word for M_4 with $\|W\| < b$. Then the time of accepting W by any reduced computation of M_4 is at most $4|W|_a + 3 \log b$.*

Proof. Let H be the reduced history of an accepting computation of M_4 with the first word W . By Lemma 4.34, the step history of H is a suffix of $(2)(1)(2)(3)$. Hence the possible Step histories are $(2)(1)(2)(3)$, $(1)(2)(3)$, $(2)(3)$ or (3) . We shall prove (a) and (b) in each of these cases.

Suppose that the step history of H is $(2)(1)(2)(3)$. Then

$$H \equiv H_2(12)^{-1}H_1(12)H'_2(23)H_3,$$

where H_2, H'_2 consist of rules of Step 2, H_1 (resp. H_3) consists of rules of Step 1 (resp. Step 3). By Lemma 4.33, the length of $(12)H'_2(23)$ is one of the T_i . Since every rule of Step 2 multiplies the tk -sector by a letter uniquely determined by that rule, and in any word in the domain of (23) , the tk -sector is empty, we conclude that the tk -sector of the word $W \cdot H_2(12)^{-1}H_1(12)$ is a copy of H'_2 , hence its length is $T_i - 2$. The $k't'$ -sector of that word is empty and every rule from H'_2 multiplies that sector by a letter uniquely

determined by the rule. Hence the $k't'$ -sector of $W \cdot H_2(12)^{-1}H_1(12)H'_2$ has length $T_i - 2$. Since every rule of Step 3 multiplies that sector by a letter, and in the stop word of M_4 that sector is empty, we conclude that $\|H_3\| = T_i - 2$. Hence $\|(12)H'_2(23)H_3\| \leq 2T_i - 2$.

Note that since every rule of Step 2 multiplies the $k't'$ -sector by a letter uniquely determined by that rule, and in a word in the domain of $(12)^{-1}$ that sector is empty, we can conclude that $\|H_2\| \leq |W|_a$. Similarly since the rules of Step 1 multiply the tk -sector by letters uniquely determined by these rules, we conclude that

$$\|H_1\| \leq \|H_2\| + |W|_a + \|H'_2\| \leq 2|W|_a + T_i - 2.$$

(We use that if a group word U of length l is obtained from a word V of lengths k after a series of one-side multiplications by one letter, and successive multiplications are not mutual inverse, then the number of multiplications does not exceed $k + l$.) Therefore $\|H\| \leq 2T_i - 2 + \|H_2\| + \|H_1\| + 1 < 3T_i + 3|W|_a$, and since in the case under consideration, we have $\|H\| \geq 9T_i$ by the condition of the lemma, it follows that $|W|_a \geq (\|H\| - 3T_i)/3 > \|H\|/6$, as required for the part (a).

Now assume that the assumption of (b) holds. Note that every rule of H_2 multiplies the $k't'$ -sector by a letter and the input p_1s_1 -sector also by at most one letter, the rules of H_1 do not touch the input sector. Therefore the input sectors of $W \cdot H_2$ and $W \cdot H_2(12)^{-1}H_1(12)$ are the same and their lengths are at most the sum of lengths of the input sector of W and the $k't'$ -sector of W . Hence the length of the input sector of $W \cdot H_2(12)^{-1}H_1(12)$ does not exceed $|W|_a \leq b$. By the condition of the lemma, we have that $\|H'_2\| = T_i - 2 \leq \log b - 2$ for some i . As before $\|H\| \leq 3T_i + 3|W|_a \leq 3|W|_a + 3 \log b$. Suppose that the step history of H is (1)(2)(3), that is $H = H_1(12)H_2(23)H_3$, where H_i contains only rules of Step i ($i = 1, 2, 3$). Then again by Lemma 4.33 $\|H_2\| = T_i - 2$ for some i , and the length of the tk -sector in $W \cdot H_1$ is $T_i - 2$. As in the previous paragraph, $\|H_3\| = T_i - 2$.

Under the assumptions of (a) then $\|H_1\| > 9T_i - 2T_i + 2 > 7T_i$. Since every rule of H_1 multiplies the tk -sector by a letter, we also have that $\|H_1\|$ does not exceed the sum of lengths of tk -sectors in W and in $W \cdot H_1$, whence $\|H_1\| \leq |W|_a + T_i - 2$. Therefore $|W|_a > 6T_i$ and

$$\|H\| = \|H_1\| + 2T_i - 2 < 2|W|_a.$$

Suppose that the assumptions of (b) hold. Then since the input sectors of W and $W \cdot H_1(12)$ are the same, and their length is $\leq |W|_a < b$, we conclude that $T_i \leq \log b$, and

$$\|H\| \leq \|H_1\| + \|H_2\| + \|H_3\| + 2 \leq |W|_a + T_i + T_i + T_i + 2 \leq 2|W|_a + 3 \log b.$$

Suppose that the step history is (2)(3), that is $H \equiv H_2(23)H_3$, and, again, H_i has rules only from Step i , $i = 2, 3$. Note that every rule of H_2 multiplies the tk -sector by a letter, and that sector in any word which is a domain of (23) must be empty. Hence $\|H_2\| \leq |W|_a$. Every rule in H_2, H_3 multiplies the $k't'$ -sector by a letter, hence $\|H_3\| \leq |W|_a + \|H_2\| \leq 2|W|_a$. Therefore $H \leq \|H_2\| + \|H_3\| + 1 \leq 3|W|_a + 1 \leq 4|W|_a$. This implies both (a) and (b).

Finally suppose that the step history is (3). Then clearly $\|H\| \leq |W|_a$, and both (a) and (b) follow.

We conclude that in every case both (a) and (b) hold.

Lemma 4.39. *Suppose that an admissible word of M_4 is accepted by M_4 , H is a history of an accepting computation. Then $\|W\|_a \leq 4\|H\|$.*

Proof. Indeed, $W \cdot H$ does not contain a -letters, and each rule of H decreases the number of a -letters in the admissible word by at most 4 (every rule of M_4 affects at most four a -letters: two letters in the subword between k and k' , one letter in the subword between t and k and one letter in the subword between k' and t'). \square

Definition 4.40. Let Q_i be a base letter. (Recall that usually we take a representative $q_i \in Q_i$.) We say that Q_i (or q_i) is *active from the left* (resp., *from the right*) for a rule θ if in the corresponding component $v_{i-1}q_iu_i \rightarrow v'_{i-1}q'_iu'_i$ of θ , the word $v_{i-1}^{-1}v_{i-1}$ (resp. $u'_iu_i^{-1}$) is not trivial (and so equal to a letter the free group by Property 3.1 (1) of M_4). If q_i is active from the left (right) for θ , then we say that q_i^{-1} is active from the right (left) for θ . We also say that q_i active for θ if it is active from the left or active from the right. Otherwise q_i is *passive* for θ .

Lemma 4.41. *Let a reduced computation of M_4 have history $(12)H$ and have the base s_0p_1 . Suppose that the letter p_1 is active in every rule θ of step 2 from H . Then every rule of H is of Step 2.*

Proof. Recall, that the rule (12) extends the start rule $\theta = \theta_{start}$ of M_3 . Therefore the first rule of H has $p^{\theta,1}(1)$ in the left-hand side. If no rule of H changes $p^{\theta,1}(1)$, then every rule is (the extension of) a rule of the machine $\vec{Z}^{\theta,1}$ with the p_1 -part of the form $p^{\theta,1}(1) \rightarrow a'p^{\theta,1}(1)(a'')^{-1}$, where p_1 is active from the both sides. Otherwise the history has a subword of type either $(12)H'(12)^{-1}$ where p_1 -part of every rule of H' is of form $p^{\theta,1}(1) \rightarrow a'p^{\theta,1}(1)(a'')^{-1}$ because the (the copy of the) rule ξ_2 of $\vec{Z}^{\theta,1}$ belongs to Step 2 but it is passive. However this case is impossible since then every rule of H' inserts (or deletes) one letter a' in the s_0p_1 -sector from the right, different rules insert different letters, and the s_0p_1 -sector is empty when the rule (12) or $(12)^{-1}$ is applicable. \square

Every rule either makes a control state letter p_i active from both sides or locks a neighbor sector. This property is useful for computations with non-standard bases as in the following

Lemma 4.42. *If the base of a reduced computation of M_4 contains a subword $(pp^{-1}p)^{\pm 1}$, where p is a control state letter, then all rules of the computation correspond to the copy of the S -machine \vec{Z} or of \overleftarrow{Z} containing that state letter, and either every rule is a copy of some $\xi_1(a)^{\pm 1}$ (a depends on the rule) or every rule is a copy of some $\xi_3(a)^{\pm 1}$.*

Proof. Indeed, suppose that $p \in P_i$. Then every rule not from the copy of \vec{Z} or of \overleftarrow{Z} containing that state letter, locks the sector $s_{i-1}p_i$ or the sector p_is_i , and the copies of ξ_2 and of ξ_4 lock either sector $s_{i-1}p_i$ or sector p_is_i . Now we can apply Lemma 3.4. \square

Lemma 4.38 and the following lemma show the role of the 'historical' tk - and $k't'$ -sectors.

Lemma 4.43. *Suppose that a reduced computation of M_4 with the standard base has the history of the form $(12)H_2(23)H_3(23)^{-1}H'_2(12)^{-1}$, where H_2, H'_2 contain rules from Step 2, and H_3 has rules of step 3. Then $\|H_3\| \leq \|H_2\| + \|H'_2\|$.*

Proof. Let W be the initial word of the computation. Since every rule of H_2, H'_2 multiplies the $k't'$ -sector by one letter which determines the rule, and every word in the domain of $(12)^{\pm 1}$ has that sector empty, we conclude that $\|H_2\|$ is equal to the length of the

$k't'$ -sector U of $W \circ (12)H_2$, and $\|H'_2\|$ is equal to the length of the $k't'$ -sector V in $W \circ (12)H_2(23)H_3$. Similarly, every rule from H_3 multiplies the $k't'$ -sector by a letter that determines the rule. Hence $\|H_3\| \leq \|U\| + \|V\| = \|H_2\| + \|H'_2\|$ as required. \square

4.4 The machine M

Consider now $2L \gg 1$ copies of the machine M_4 , denote them by $M_4(i)$, $i = 1, \dots, 2L$. We denote the state and tape letter of $M_4(i)$ accordingly, by adding index i to all letters, and all rules. Let $\Theta(M_4)$ be the set of positive rules of M_4 . Let B be the standard base of M_3 , $B(i)$ be the copy word B with new extra index i added to all letters, and $B_i = k(i)B(i)k'(i)$. We now consider the S -machine M with the rules

$$\theta(M) = [\theta(1), \dots, \theta(2L)], \quad \theta \in \Theta(M_4)$$

(we shall denote $\theta(M)$ by θ also) and the standard base

$$t_1 B_1 t'_2 B_2^{-1} t_3 B_3 t'_4 B_4^{-1} \dots t'_{2L} B_{2L}^{-1} t_{2L+1}, \quad (4.6)$$

where we identify the state t -letters $t(1)$ and $t'(1)$ of $M_4(1)$ with t_1 and t_2 , resp., the t -letters $t(2)$ and $t'(2)$ of $M_4(2)$ with t_3^{-1} and $(t'_2)^{-1}$, resp, and so on. Moreover we identify t_{2L+1} with t_1 and consider the standard base of M up to cyclic permutations which may start with any t -letter and end with the same t -letter. The stop word is defined accordingly (every letter in the standard base is replaced by the corresponding letter in the stop word of $M_4(i)$). The stop word without the last letter t_{2L+1} is called the *hub*. We also may take the hub up to cyclic permutations.

That construction is similar to the construction in [19] and [14], though the application of mirror copies of machines goes back to the works of Boon and P.S. Novikov (see [17]). The condition $L \gg 1$ makes hub graph hyperbolic (see Lemmas 5.18 and 5.19), and the mirror symmetry of the word (4.6) is used for the surgery we define in Subsection 12.2.

For every admissible word W of M_4 with the standard base we denote by $W(M)$ the corresponding admissible word $t_1 k(1)W(1)k'(1)t'_2 k'(2)^{-1}W(2)^{-1}k(2)^{-1}t_2 \dots$ of M with the standard base (of M). By definition, $W(M)$ is an input (the accept) word of M if W is an input (the accept) word of M_4 .

The letters in the copy $W(i)$ of the word W are equipped with the extra index (i) . Thus every a -letters and every q -letter (except for t and t' -letters), and every letter $\theta(i)$ of the alphabets of M has this extra index. We call it the M -*index* of the letter and take it modulo $2L$.

Remark 4.44. (1) Notice that for every rule θ of M_4 and every admissible word W of M_4 with the standard base of M_4 , we have $W \cdot \theta = W'$ if and only if $W(M) \cdot \theta = W'(M)$.

(2) Also notice that $\|W(M)\| < 2L\|W\|$ for every W .

(3) Both machines M_4 and M enjoy Property 3.1 (1) but not Property 3.1 (2).

(4) The unique start and accept rules of the machines M_1 , M_2 , M_3 are converted to the transition rules (12) and (34) of M_4 and M . So there are no specific start and accept rules of M_4 and M . In particular M accepts if it reaches the hub.

Remark 4.44 (1) and Lemma 4.36 immediately imply

Lemma 4.45. *Let X_5 be the set of all words of the form $W(M)$, $W \in X_4$. Then for every input word W of M_4 , W is accepted by M_4 if and only if $W(M)$ is accepted by M and if and only if $W \in X_4$. Hence the set X_5 is not recursive.*

Remark 4.46. Considered as a cyclic word, the hub has the following symmetries: it does not change if we reflect it about any t -letter or any t' -letter (with indices changing appropriately, and state letters, except for t - and t' -letters, replaced by their inverses). From Remark 4.44 (1), it follows, that every admissible accepted word of M has similar symmetries.

5 Groups and diagrams

Every S-machine can be considered as a finitely presented group (see [19] and also [15], [13]). Here we apply the construction to the machine M . To simplify formulas, it is convenient to redefine N once again. From now on we shall denote by $N + 1$ the length of the smallest subword of the hub containing two t -letters. Thus the length of the hub is LN , $Q = \sqcup_{i=0}^{LN} Q_i$ (where $Q_{LN} = Q_0$) $Y = \sqcup_{i=1}^{LN} Y_i$, and Θ is the set of rules of the S-machine M . (But we will remember that, as for the machine M_4 , the state letter of M are partitioned into the subsets of t -letters, t' -letters, k -letters, k' -letters, s -letters, and p -letters.)

The finite set of generators of the group M (the same letter as for the machine) consists of q -letters corresponding to the states Q , a -letters corresponding to the tape letters from Y , and θ -letters corresponding to the rules from the positive part Θ^+ of Θ .

The relations of the group M correspond to the rules of the machine M ; for every $\theta = [U_0 \rightarrow V_0, \dots, U_{LN} \rightarrow V_{LN}] \in \Theta^+$, we have

$$U_i \theta_{i+1} = \theta_i V_i, \quad \theta_j a = a \theta_j, \quad i, j = 0, \dots, LN \quad (5.7)$$

for all $a \in \bar{Y}_j(\theta)$. (Here $\theta_{LN} \equiv \theta_0$.) The first type of relations will be called (θ, q) -relations, the second type - (θ, a) -relations.

Finally, the required group G is given by the generators and relations of the group M and by one more additional relation, namely the *hub*-relation

$$W_M = 1, \quad (5.8)$$

where W_M is the hub, i.e., the accept word (of length LN) of the machine M .

Remark 5.1. The word W_M has the symmetries mentioned in Remark 4.46. Since the machine M is built of $2L$ copies $M_4(i)$, the set of relations is also symmetric in the following sense. Every relation $\theta_j a = a \theta_j$ from (5.7) has $2L$ copies (including itself) corresponding to different $M_4(i)$ -s. If the relation $U_i \theta_{i+1} = \theta_i V_i$ from (5.7) involves neither t - nor t' -letters then it has L copies (including itself) and L mirror copies. Every relation containing a t - or a t' -letter (denote this letter by \tilde{t}) has form $\tilde{t} \theta_{i+1} = \theta_i \tilde{t}$, i.e., it contains no a -letters.

5.1 Minimal diagrams

Recall that a van Kampen *diagram* Δ over a presentation $P = \langle A \mid \mathcal{R} \rangle$ (or just over the group P) is a finite oriented connected and simply-connected planar 2-complex endowed with a labeling function $\text{Lab} : E(\Delta) \rightarrow A^{\pm 1}$, where $E(\Delta)$ denotes the set of oriented edges of Δ , such that $\text{Lab}(e^{-1}) \equiv \text{Lab}(e)^{-1}$. Given a cell (that is a 2-cell) Π of Δ , we denote by $\partial \Pi$ the boundary of Π ; similarly, $\partial \Delta$ denotes the boundary of Δ . The labels of $\partial \Pi$ and $\partial \Delta$ are defined up to cyclic permutations. An additional requirement is that the

label of any cell Π of Δ is equal to (a cyclic permutation of) a word $R^{\pm 1}$, where $R \in \mathcal{R}$. The label and the combinatorial length $\|\mathbf{p}\|$ of a path \mathbf{p} are defined as for Cayley graphs.

The van Kampen Lemma states that a word W over the alphabet $A^{\pm 1}$ represents the identity in the group P if and only if there exists a diagram Δ over P such that $\text{Lab}(\partial\Delta) \equiv W$, in particular, the combinatorial perimeter $\|\partial\Delta\|$ of Δ equals $\|W\|$. ([7], Ch. 5, Theorem 1.1). The word W representing 1 in P is freely equal to a product of conjugates to the words from $R^{\pm 1}$. The minimal number of factors in such products is called the *area* of the word W . The *area* of a diagram Δ is the number of cells in it. A diagram having the smallest number of cells among all diagrams with the same boundary label is called *minimal*. By van Kampen's Lemma, $\text{Area}(W)$ is equal to the area of a minimal diagram Δ over P with $\text{Lab}(\partial\Delta) \equiv W$. This definitions imply

Lemma 5.2. *Assume that a diagram Δ_0 is divided into two subdiagrams Δ_1 and Δ_2 by a simple path p . Let a minimal diagram Δ have the same boundary label as Δ_0 . Then $\text{Area}(\Delta) \leq \text{Area}(\Delta_0) = \text{Area}(\Delta_1) + \text{Area}(\Delta_2)$.*

We will study diagrams over the groups M and G . The edges labeled by state letters ($= q$ -letters) will be called q -edges, the edges labeled by tape letters ($= a$ -letters) will be called a -edges, and the edges labeled by θ -letters are θ -edges.

Remark 5.3. The symmetries of relations observed in Remark 5.1 makes possible the following construction for given $i \leq 2L$ and a diagram Δ over M . Let ∇ be a mirror copy of the map Δ . For every edge e of Δ whose label is equipped with an M -index (j) (i.e., if e is neither t - nor t' -edge), the mirror copy of e in ∇ is marked by the same letter but with M -index equal to $(2i - j - 1)$. The label t_j of e (the label t'_j) should be replaced for the mirror image by t_{2i-j}^{-1} (resp., by $(t'_{2i-j})^{-1}$). It is easy to see that ∇ is also a diagram over M . We say that ∇ is obtained by t_i -reflection from Δ . Similarly one can speak on t_i -reflections for paths of Δ .

We denote by $|\mathbf{p}|_a$ (by $|\mathbf{p}|_\theta$, by $|\mathbf{p}|_q$) the a -length (resp., the θ -length, the q -length) of a path/word \mathbf{p} , i.e., the number of a -edges/letters (the number of θ -edges/letters, the number of q -edges/letters) in \mathbf{p} .

The cells corresponding to Relation (5.8) are called *hubs*, the cells corresponding to (θ, q) -relations are called (θ, q) -cells, and they are called (θ, a) -cells if they correspond to (θ, a) -relations.

Every minimal van Kampen diagram is *reduced*, that is it does not contain two cells ($=$ closed 2-cells) that have a common edge and are mirror images of each other (if such pairs of cells exist, they can be removed to obtain a diagram of smaller area and with the same boundary label). To study (van Kampen) diagrams over the group G we shall use their simpler subdiagrams such as bands and trapezia, as in [11], [19], [1], etc. Here we repeat one more necessary definition.

Definition 5.4. Let \mathcal{Z} be a subset of the set of generators \mathcal{X} of the group M . A \mathcal{Z} -band \mathcal{B} is a sequence of cells π_1, \dots, π_n in a reduced van Kampen diagram Δ such that

- Every two consecutive cells π_i and π_{i+1} in this sequence have a common edge e_i labeled by a letter from \mathcal{Z} .
- Each cell π_i , $i = 1, \dots, n$ has exactly two \mathcal{Z} -edges, e_{i-1} and e_i (i.e. edges labeled by a letter from \mathcal{Z}).

- If $n = 0$, then \mathcal{B} is just a \mathcal{Z} -edge.

The counterclockwise boundary of the subdiagram formed by the cells π_1, \dots, π_n of \mathcal{B} has the factorization $e^{-1}\mathbf{q}_1 f \mathbf{q}_2^{-1}$ where $e = e_0$ is a \mathcal{Z} -edge of π_1 and $f = e_n$ is an \mathcal{Z} -edge of π_n . We call \mathbf{q}_1 the *bottom* of \mathcal{B} and \mathbf{q}_2 the *top* of \mathcal{B} , denoted $\mathbf{bot}(\mathcal{B})$ and $\mathbf{top}(\mathcal{B})$. Top/bottom paths and their inverses are also called the *sides* of the band. The \mathcal{Z} -edges e and f are called the *start* and *end* edges of the band. If $n \geq 1$ but $e = f$, then the \mathcal{Z} -band is called a \mathcal{Z} -annulus.

We will consider q -bands, where \mathcal{Z} is one of the sets Q_i of state letters for the machine M , θ -bands for every $\theta \in \Theta$, and a -bands, where $\mathcal{Z} = \{a\} \subseteq Y$. The convention is that a -bands do not contain (θ, q) -cells, and so they consist of (θ, a) -cells only.

Remark 5.5. To construct the top (or bottom) path of a band \mathcal{B} , at the beginning one can just form a product $\mathbf{x}_1 \dots \mathbf{x}_n$ of the top paths \mathbf{x}_i -s of the cells π_1, \dots, π_n (where each π_i is a \mathcal{Z} -bands of length 1). No θ -letter is being canceled in the word $W \equiv \text{Lab}(x_1) \dots \text{Lab}(x_n)$ if \mathcal{B} is a q - or a -band since otherwise two neighbor cells of the band would be mirror copies of each other which is impossible in a reduced diagram.

Also there are no cancellations of a -letters if \mathcal{B} is a q -band. Indeed if both π_i and π_{i+1} have a -edges on their top then the corresponding rules of M must belong to the same Step since every cell is passive for (12)- and (23)-rules. Similarly they correspond to the rule of the same machine $\overrightarrow{Z}^{(\theta, i)}$ or $\overleftarrow{Z}^{(\theta, i)}$ if \mathcal{B} is a q -band for some control letter p_i since the rules ξ_2 and ξ_4 provide no active cells. Then the rules are determined by the a -letters, and the cells should be mirror copies as in the previous paragraph. Similar argument works if q corresponds to any other letter of the standard base except for s_i . But active s_i -cell cannot have a common edge too since this edge has a θ -index in the label, and so the diagram is not reduced again.

Thus, if \mathcal{B} is a q -band (or an a -band), then the top/bottom label is a product $\mathbf{x}_1 \dots \mathbf{x}_n$. If \mathcal{B} is a θ -band then a few cancellations of a -letters (but not θ -letters) are possible in W . (This can happen if one of π_i, π_{i+1} is a (θ, q) -cell and another one is a (θ, a) -cell.) We will always assume that the top/bottom label of a θ -band is a reduced form of the word W . This property is easy to achieve: by folding edges with the same labels having the same initial vertex, one can make the boundary label of a subdiagram in a van Kampen diagram reduced (e.g., see [19]).

If the path $(e^{-1}\mathbf{q}_1 f)^{\pm 1}$ or the path $(f\mathbf{q}_2^{-1}e^{-1})^{\pm 1}$ is the subpath of the boundary path of Δ then the band is called a *rim* band of Δ . We shall call a \mathcal{Z} -band *maximal* if it is not contained in any other \mathcal{Z} -band. Counting the number of maximal \mathcal{Z} -bands in a diagram we will not distinguish the bands with boundaries $e^{-1}\mathbf{q}_1 f \mathbf{q}_2^{-1}$ and $f\mathbf{q}_2^{-1}e^{-1}\mathbf{q}_1$, and so every cell having two \mathcal{Z} -edges belongs to a unique maximal \mathcal{Z} -band.

We say that a \mathcal{Z}_1 -band and a \mathcal{Z}_2 -band *cross* if they have a common cell and $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset$.

Sometimes we specify the types of bands as follows. A θ -band corresponding to the transition rule (12) (to (23)) is called a (12)-band ((23)-band), and it consists of (12)-cells (of (23)-cells). A q -band corresponding to one of the letters t_i of the base (4.6) (resp., to $t'_i, k(i), k'(i)$) is called a t -band (t' -band, k -band, k' -band) since the M -index i is, generally, not important for further considerations (but we may keep it if it is essential). Similarly, we can omit the M -index speaking on s - and p -bands, but we distinguish different letters of each particular B_i in the standard base, e.g., the s_0 -letter follows after the k -letter in each subword B_i hence the standard base (4.6) has $2L$ different s_0 -letters (one in each

subword B_i), $2L$ different p_1 -letters, and so on. Also this agreement allows to speak on (12) -letters and (12) -edges, \dots , p_i -letters, p_i -edges, and p_i - (or s_i -) bands.

The papers [12], [1], [16] contain the proof of the following lemma in a more general setting. (In contrast to Lemmas 6.1 [12] and 3.11 [16], we have no x -cells here.)

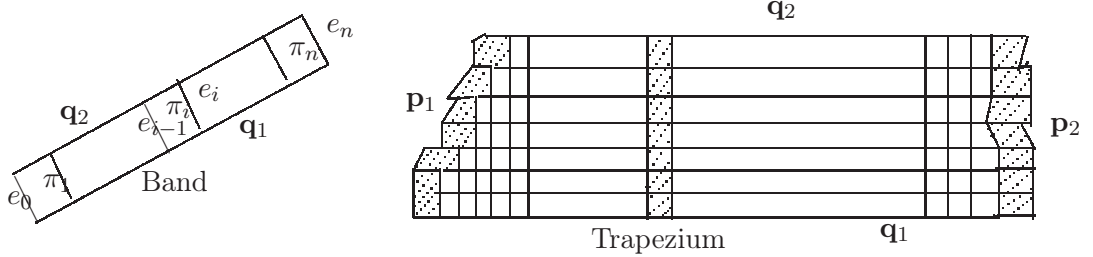
Lemma 5.6. *A reduced van Kampen diagram Δ over M has no q -annuli, no θ -annuli, and no a -annuli. Every θ -band of Δ shares at most one cell with any q -band and with any a -band.*

If $W \equiv x_1 \dots x_n$ is a word in an alphabet X , X' is another alphabet, and $\phi: X \rightarrow X' \cup \{1\}$ (where 1 is the empty word) is a map, then $\phi(W) \equiv \phi(x_1) \dots \phi(x_n)$ is called the *projection* of W onto X' . We shall consider the projections of words in the generators of M onto Θ (all θ -letters map to the corresponding element of Θ , all other letters map to 1), and the projection onto the alphabet $\{Q_0 \sqcup \dots \sqcup Q_{LN-1}\}$ (every q -letter maps to the corresponding Q_i , all other letters map to 1).

Definition 5.7. The projection of the label of a side of a q -band onto the alphabet Θ is called the *history* of the band. The Step history of this projection is the *Step history* of the q -band. The projection of the label of a side of a θ -band onto the alphabet $\{Q_0, \dots, Q_{LN-1}\}$ is called the *base* of the band, i.e., the base of a θ -band is equal to the base of the label of its top or bottom.

As for words, we will use representatives of Q_j -s in base words. (If $p \in Q_4$, $s \in Q_5$, we shall say that the word pas has base ks instead of Q_4Q_5 , and so on.)

Definition 5.8. Let Δ be a reduced diagram over M which has boundary path of the form $\mathbf{p}_1^{-1} \mathbf{q}_1 \mathbf{p}_2 \mathbf{q}_2^{-1}$, where \mathbf{p}_1 and \mathbf{p}_2 are sides of q -bands, and \mathbf{q}_1 , \mathbf{q}_2 are maximal parts of the sides of θ -bands such that $\text{Lab}(\mathbf{q}_1)$, $\text{Lab}(\mathbf{q}_2)$ start and end with q -letters.



Then Δ is called a *trapezium*. The path \mathbf{q}_1 is called the *bottom*, the path \mathbf{q}_2 is called the *top* of the trapezium, the paths \mathbf{p}_1 and \mathbf{p}_2 are called the *left and right sides* of the trapezium. The history (Step history) of the q -band whose side is \mathbf{p}_2 is called the *history* (resp., Step history) of the trapezium; the length of the history is called the *height* of the trapezium. The base of $\text{Lab}(\mathbf{q}_1)$ is called the *base* of the trapezium.

Remark 5.9. Notice that the top (bottom) side of a θ -band \mathcal{T} does not necessarily coincides with the top (bottom) side \mathbf{q}_2 (side \mathbf{q}_1) of the corresponding trapezium of height 1, and \mathbf{q}_2 (\mathbf{q}_1) is obtained from $\mathbf{top}(\mathcal{T})$ (resp. $\mathbf{bot}(\mathcal{T})$) by trimming the first and the last a -edges if these paths start and/or end with a -edges. We shall denote the *trimmed* top and bottom sides of \mathcal{T} by $\mathbf{ttop}(\mathcal{T})$ and $\mathbf{tbot}(\mathcal{T})$. By definition, for arbitrary θ -band \mathcal{T} , $\mathbf{ttop}(\mathcal{T})$ is obtained by such a trimming only if \mathcal{T} starts and/or ends with a (θ, q) -cell; otherwise $\mathbf{ttop}(\mathcal{T}) = \mathbf{top}(\mathcal{T})$. The definition of $\mathbf{tbot}(\mathcal{T})$ is similar.

By Lemma 5.6, any trapezium Δ of height $h \geq 1$ can be decomposed into θ -bands $\mathcal{T}_1, \dots, \mathcal{T}_h$ connecting the left and the right sides of the trapezium. The word written on the trimmed top side of one of the bands \mathcal{T}_i is the same as the word written on the trimmed bottom side of \mathcal{T}_{i+1} , $i = 1, \dots, h$. Moreover, the following lemma claims that every trapezium simulates the work of M . It summarizes the assertions of Lemmas 6.1, 6.3, 6.9, and 6.16 from [14]. For the formulation (1) below, it is important that M is an S -machine. The analog of this statement is false for Turing machines. (See [13] for a discussion.)

Lemma 5.10. (1) *Let Δ be a trapezium with history $\theta_1 \dots \theta_d$ ($d \geq 1$). Assume that Δ has consecutive maximal θ -bands $\mathcal{T}_1, \dots, \mathcal{T}_d$, and the words U_j and V_j are the trimmed bottom and the trimmed top labels of \mathcal{T}_j , ($j = 1, \dots, d$). Then U_j, V_j are admissible words for M , and*

$$V_1 \equiv U_1 \cdot \theta_1, U_2 \equiv V_1, \dots, U_d \equiv V_{d-1}, V_d \equiv U_d \cdot \theta_d$$

(2) *For every reduced computation $U \rightarrow \dots \rightarrow U \cdot H \equiv V$ of M with $\|H\| \geq 1$ there exists a trapezium Δ with bottom label U , top label V , and with history H .*

If $H' \equiv \theta_i \dots \theta_j$ is a subword of the history $\theta_1 \dots \theta_d$ from Lemma 5.10 (1), then the bands $\mathcal{T}_i, \dots, \mathcal{T}_j$ form a subtrapezium Δ' of the trapezium Δ . This subtrapezium is uniquely defined by the subword H' (more precisely, by the occurrence of H' in the word $\theta_1 \dots \theta_d$), and Δ' is called the H' -part of Δ .

5.2 Properties of the group M .

In this subsection, we want to translate the properties of the machine M in the language of diagrams over the group M .

Recall that every (θ, q) -cell π has a boundary label of the form $U_i \theta_{i+1} V_i^{-1} \theta_i^{-1}$ (see Relations (5.7)), where the word U_i (the word V_i) has exactly one positive q -letter $q_i \in Q_i$ ($q'_i \in Q_i$). Hence the boundary label of π is $q_i w_1 (q'_i)^{-1} w_2$ for some words w_1, w_2 .

Definition 5.11. The cell π considered as a one-cell q -band with base q_i is called *active from the right (from the left)* if the word w_1 (the word w_2) has at least one a -letter. If π with base q_i is active from the right (from the left) then, by definition, the same cell considered as a q -band with base q_i^{-1} is active from the left (resp., from the right). A (θ, q) -cell is called *passive* if it is not active either from the left or from the right.

The comparison with Definition 4.40 shows that the cell π with base $q_i^{\pm 1}$ is active from the left (resp., active from the right, passive) iff the base letter $q_i^{\pm 1}$ is active on the left (resp., active on the right, passive) for the rule corresponding to the θ -edges of π .

Definition 5.12. We say that a q -band with base q_i is *active from the left (from the right)* if every cell of it (with the same base) except for the first cell and the last one (if the first and/or the first cell corresponds to the rules $(12)^{\pm 1}$ or $(23)^{\pm 1}$), is active from the left (from the right). A q -band is called *passive* if every its cell is passive. Similarly one can speak on a q -band with base q_i which is *passive from the left* or *passive from the right*.

Remark 5.13. The letter s_0 in the standard base of M_4 corresponds to the left-most α -marker of the machines $M_1 - M_4$, and so every s_0 -band is passive (from both sides).

Definition 5.14. We say that a q -band \mathcal{C} with base q_i is *strongly active from the left* (resp. *right*) if every its cell π is active from the left (from the right), $\partial\pi$ has exactly one a -edge on the left side (right side) of \mathcal{C} , and these a -letters are different for the cells corresponding to different rules of the history of \mathcal{C} .

Lemma 5.15. *Let Δ be any reduced diagram over M . Let \mathcal{C} be a q -band, corresponding to the part Q_i of Q . Suppose that \mathcal{C} is strongly active from the left (resp. right). Then Δ does not have an a -band starting and ending on the left side (resp., right side) of \mathcal{C} .*

Proof. Suppose that an a -band \mathcal{A} starts and ends on $\mathbf{bot}(\mathcal{C})$ which is the left side of \mathcal{C} . Let Δ' be the subdiagram bounded by \mathcal{A} and \mathcal{C} .

Then Δ' has no other maximal q -bands except the part $\mathcal{C}' = \mathcal{C} \cap \Delta'$ because a -bands and q -bands do not intersect and Δ' has no q -annuli by Lemma 5.6. Since maximal a -bands do not intersect, we can assume without loss of generality that Δ' does not have any other a -bands starting and ending on $\mathbf{bot}(\mathcal{C}')$. Since \mathcal{C}' is strongly active on the left, and the sides of \mathcal{A} consist of θ -edges, we conclude that \mathcal{C}' consists of two cells having common q - and a -edges. A θ -cell in \mathcal{C} is completely determined by its a -letter on its bottom side (see Remark 5.5 for the argument). Therefore those two q -cells cancel, a contradiction with the assumption that Δ is reduced. \square

The next Proposition summarizes previously proved properties of the various submachines of the S -machine M . We formulate these properties in the language of van Kampen diagrams which makes it more convenient to apply these properties to the group G .

Recall that the standard base of M_3 is denoted by B . Note that the standard base of M contains L copies $B(i)$ of B ($i = 1, 3, \dots, 2L - 1$) and L copies of B^{-1} . We call the base of an admissible word of M *aligned*, if every maximal subword of this base without letters t, k, k', t' is a subword of a copy of $B^{\pm 1}$.

Remark 5.16. Since $\|B\| < N/2$, Formula 4.6 and the definition of admissible words show that every aligned base of length $\geq N/2$ must contain a $k^{\pm 1}$ - or a $(k')^{\pm 1}$ -letter or entirely consists of $t^{\pm 1}$ - or $(t')^{\pm 1}$ -letters.

A base of an admissible word of M is called *normal* if it is a subword of a power of the base of the hub. (Recall that t_1 and t_{2L+1} were identified in the definition of the machine M .) A base is called *large* if it contains a copy of $B^{\pm 1}$.

We shall say that a (θ, q) -cell π in a van Kampen diagram over M is *odd* if it contains exactly one a -edge on its boundary, and its base is not k or k' . A θ -band with (1-letter) history of type (2) is called *odd* if it contains odd cells.

A trapezium over M whose top label is one of $2L$ copies of the stop word of M_4 will be called *M_4 -accepting trapezium*. (The trapezium pictured in Subsection 4.3 is M_4 -accepting and in addition, its bottom label is an input word of M_4 .) A trapezium whose base is a copy of the standard base of M_4 is *standard* if its bottom label is in the domain of the rule (12) and its top label is in the domain of $(23)^{-1}$. By Lemmas 4.33 and 5.10, every standard trapezium has height T_i for some i and corresponds to a standard computation of M_4 .

Proposition 5.17. *The following properties of the group M hold. In all these properties we assume that we are given a reduced van Kampen diagram Δ over M , all bands, cells and edges are bands, cells and edges of that diagram.*

- (i) A two letter base of a θ -band is either a subword of the word (4.6) or of the inverse word, or it has form $q^{\pm 1}q^{\mp 1}$ for a base letter q .
- (ii) Every cell with base k (every cell with base k') corresponding to a rule of Step 1 or Step 2 except for the $(12)^{\pm 1}$ -rule (corresponding to a rule of Step 2 or Step 3 except for the $(23)^{\pm 1}$ -rule), is active from the left (resp., from the right) and passive from the right (resp., from the left). Every non- k -cell (non- k' -cell) corresponding to a rule of Step 1 (resp., of Step 3) is passive.
Every t -, t' -, and s_0 -band is passive.
- (iii) (a) The boundary of every cell has at most two a -edges. It has either 0 or 2 a -edges if it is a (θ, q) -cell corresponding to a control letter p_i , otherwise it has at most one a -edge.
(b) If there are two a -letters a and a' in the boundary label of a cell π then a' is a copy of a^{-1} and π is either a (θ, a) -cell or a (θ, q) -cell corresponding to a control letter p_i , and the a -edges are separated by q -edges in $\partial\pi$.
(c) A (θ, a) -cell has two mutually inverse θ -letters in the boundary label.
(d) Two (θ, q) -cells corresponding to control letters p_i and p_j with $i \neq j$, have no common a -letter in the boundary labels.
- (iv) If a θ -band \mathcal{T} has 3 consecutive cells π_1, π_2, π_3 , where π_2 corresponds to a control letter $p_i^{\pm 1}$ and π_2 is not active from both sides then one of the cells π_1, π_3 is a (θ, q) -cell whose base is an s -letter.
- (v) Let a (θ, q) -cell π_i of a θ -band \mathcal{T} have base q . If \mathcal{T} corresponds to a rule of Step 1 or to (12) (respectively, Step 3 or to (23)), and the next cell, π_{i+1} in \mathcal{T} is a (θ, a) -cell, then q can be only one of the following letters: t, p_1, k^{-1}, s_1^{-1} (resp., k' or $(t')^{-1}$) (with some indices). For other values of q the next after q letter in the base of \mathcal{T} cannot be q^{-1} .
- (vi) If in the base of a θ -band, there is a subword $p_i^{\pm 1}p_i^{\mp 1}p_i^{\pm 1}$ for some control letter p_i and there are neither $k^{\pm 1}$ - nor $(k')^{\pm 1}$ -letters, then the active cells in this band are precisely the p_i -cells, and these cells are active from both sides.
- (vii) Suppose that \mathcal{C} is a $k^{\pm 1}$ -, $(k')^{\pm 1}$ -, or $p_i^{\pm 1}$ -band with top path \mathbf{y} . Suppose that each cell of \mathcal{C} has a common a -edge with \mathbf{y} . Then no a -band of Δ can start and end on \mathbf{y} .

In the remaining parts of the Proposition, Δ is a trapezium.

- (viii) If Δ has base $k't'$ (base tk) then it cannot have Step history $(12)(2)(12)^{-1}$ (resp., $(23)^{-1}(2)(23)$).
- (ix) If $p_1p_1^{-1}s_0^{-1}$ is a subword of the base of Δ , and the history of Δ has the form $(12)H$, then H is of type (2), it has no rules $(12)^{\pm 1}$, $(23)^{\pm 1}$, and in the H -part of Δ , all p_1 -cells are active both from the left and from the right.
- (x) Suppose that Δ is M_4 -accepting. Then the step history of Δ is a subword of $(2)(1)(2)(3)$. If W is the label of the bottom path of Δ , and h is the height of Δ , then $\|W\|_a \leq 4h$.

- (xi) If the history of Δ contains $(12)^{\pm 1}$ and $(23)^{\pm 1}$, then the base of Δ is normal.
- (xii) If (a) the length of the base of Δ is at least N and its history contains both a rule $(12)^{\pm 1}$ and a rule $(23)^{\pm 1}$, or (b) the base of Δ is standard, then the step history of Δ is a subword of $(2)(1)(2)(3)(2)(1)(2)$.
- (xiii) Suppose that the base of Δ is not aligned and the history is of type (2). Then the label of the top (and of the bottom) of every maximal s_j -band of Δ admits a factorization of the form $u(b_1v_1b_1^{-1}) \dots (b_mv_mb_m^{-1})w$ where $b_i^{\pm 1}$ is an a -letter or 1 ($i = 1, \dots, m$), v_i is a group word in θ -letters, b_i commutes with every letter of v_i by virtue of (θ, a) -relations, and each of u, w has at most one a -letter.
- (xiv) If the base of Δ is large, and its history has the form H^3 , then Δ does not have odd cells π .
- (xv) Suppose that the base of Δ has the form $k^{-1}k$ or $k'(k')^{-1}$, and all k - (resp. k' -) cells of Δ are active. Let W and W' be the labels of the bottom and top of Δ respectively. Then the history of Δ has the form $H_1H_2^kH_3$, where $k \geq 0$, $\|H_1\| \leq \|W\|_a/2$, $\|H_2\| \leq \min(|W|_a, |W'|_a)$, $\|H_3\| \leq \|W'\|_a/2$.
- (xvi) If the base of Δ is of length $\geq N$, and Δ has the step history

$$(12)(2)(23)(3)(23)^{-1}(2)(12)^{-1},$$

then the height of the $(23)(3)(23)^{-1}$ -part of Δ is less than the sum of heights of the $(12)(2)$ - and $(2)(12)^{-1}$ -parts of it.

- (xvii) Let $m > 0$ be an integer such that for every standard trapezium with a bottom label W , inequality $|W|_a \leq m$ implies $\|H\| < \log m$. Suppose that Δ is M_4 -accepting, the history of Δ is H , and the bottom label W' satisfies the inequality $|W'|_a \leq m$. Then we have

$$\|H'\| \leq 4|W'|_a + 3 \log m.$$

- (xviii) The set of numbers m satisfying the assumption from (xvii) is infinite.
- (xix) Suppose Δ is M_4 -accepting. If the height h of Δ exceeds $6|W|_a$, where W is the bottom label of Δ , then there exists a standard subtrapezium Δ' in Δ such that $h \in (h', 9h')$, where h' is the height of Δ' .

□

- (i) This follows from the definition of admissible word and from Lemma 5.10.
- (ii),(iii) This follows from the definition of the rules of M , the definitions of Relations (5.7), and from Remark 5.13.
- (iv) Indeed, if a component $p_i \rightarrow \dots$ of a rule from M is not active from both sides, then it locks either $s_{i-1}p_i$ -sector or the p_is_i -sector, and we can apply Property (i) and Lemma 3.4.
- (v) Indeed, the rules from Step 1 and the rule (12) (resp. Step 3 and the rule (23)) lock all sectors except the tk -sectors, and p_1s_1 -sectors (resp. the $k't'$ -sectors) of the admissible words of M . It remains to use Lemma 3.4.
- (vi) This also follows from Lemma 3.4: if a rule of M does not lock the p_is_i -sectors or $s_{i-1}p_i$ -sectors, then its component involving p_i has the form $p_i \rightarrow ap'_ib$, where a, b are

tape letters, and all other components, except for k - and k' -components, do not involve tape letters.

(vii) The condition means that the band \mathcal{C} is active on the left and has no (passive) (12)-or (23)-cells. It follows from the definition of M that then $k^{\pm 1}, (k')^{\pm 1}$ or p_i -band is strongly active on the left. It remains to apply Lemma 5.15.

(viii) This follows from Lemmas 4.32 (a,b) and 5.10.

(ix) Let us apply Lemma 5.10 and consider the reduced computation corresponding to Δ . The rule (12) switches on the copy of the machine $\vec{Z}^{\theta,1}$ where $\theta = \theta_{start}$ is the start rule of M_3 . The (copies of the) rules of the form $\zeta_1(a)^{\pm 1}$ cannot follow by the (copy of the) rule ζ_2 since ζ_2 locks the p_1s_1 -sector. Also, by Lemma 3.5, it cannot follow by the rules (12) $^{\pm 1}$ or (23) $^{\pm 1}$ locking the s_0p_1 -sector. Therefore each of the rules of H is of the form $\zeta_1(a)^{\pm 1}$, and the statement follows.

(x) The first statement follows from Lemma 4.34 (b) because Δ is the trapezium corresponding to an accepting computation of a copy of the machine M_4 . The second property immediately follows from Lemma 4.39.

(xi) Indeed, every sector of the standard base of M is locked by either (12) or (23). It remains to use (i) and Lemma 3.4.

(xii) Indeed, by Property (xi), the base of Δ is normal. Since its length is at least N , it must contain a copy of the base of M_4 , and it remains to use Lemma 4.34 (a).

(xiii) The base has non-aligned subword B_0 without k - and t -letters. Hence the copy of $B_0^{\pm 1}$ is not a subword of the standard base of the machine M_3 . If H is the history of the corresponding computation of M_3 , then by Lemma 4.28 (2), we have $\Pi_{32}(H) \equiv (\theta^{-1})(\theta_1\theta_1^{-1}) \dots (\theta_m\theta_m^{-1})(\theta')$ for some positive rules $\theta, \theta_1, \dots, \theta'$ of \tilde{M}_2 (θ and/or θ' may be absent).

Recall that a (θ, s_j) -cell has at most one a -edge, and it has no a -edges, if it corresponds to a rule of one of the auxiliary machines $\vec{Z}, \overleftarrow{Z}$. Hence the label of a side of the s_j -band has form $u(b_1v_1b_1^{-1}) \dots (b_mv_mb_m^{-1})w$ where $b_i^{\pm 1}$ is an a -letter or 1 ($i = 1, \dots, m$), v_i is a group word in θ -letters, and each of u, w has at most one a -letter; and we should prove that b_i commutes with v_i if b_i is involved in the rule $\bar{\theta}_i$.

Let us consider the right side of the s_j -band. (The ‘left’ case is similar.) Then θ_i is a right rule, and by Lemma 4.28 (3), no rule of the subword $\theta_i H' \theta_i^{-1}$ of H locks the $s_j p^{j+1}$ -sector, and so the letter b_i commutes with every θ -letter of v_i by the definition of relations for the machine M .

(xiv) follows from Lemma 4.29 since the cells corresponding to M_3 -rules can have exactly one a -edge in the boundary only if they correspond to the rules of \tilde{M}_2 .

(xv) follows from Lemma 3.7.

(xvi) follows from Lemma 4.43 because by Property (xi) the base of the trapezium contains (as a subword) a copy of the base of M_4 .

(xvii) follows from Lemma 4.38 (b).

(xviii) follows from Lemmas 4.26 and 4.33.

(xix) This is a reformulation of Lemma 4.38 (a). □

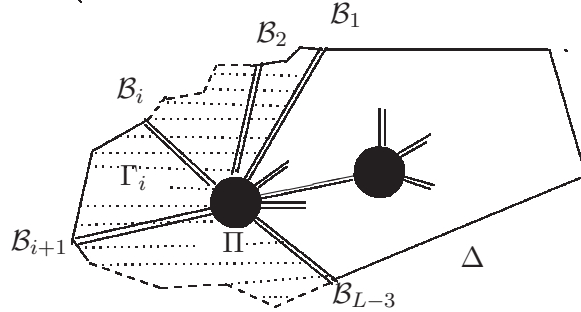
5.3 Diagrams with hubs

Given a reduced diagram Δ over the group G , one can construct a planar graph whose vertices are the hubs of this diagram plus one improper vertex outside Δ , and the edges are maximal t -bands of Δ .

Let us consider two hubs Π_1 and Π_2 in a minimal diagram, connected by a t_i -band \mathcal{C}_i and a t_{i+1} -band \mathcal{C}_{i+1} , where there are no other hubs between these t -bands. These bands, together with parts of $\partial\Pi_1$ and $\partial\Pi_2$, bound either a subdiagram having no cells, or a trapezium Ψ of height ≥ 1 . The former case is impossible since in this case the hubs have a common t -edge and they are mirror copies of each other contrary to the reducibility of the diagram. We want to show that the latter case is not possible either.

Indeed, in the latter case, both the top and the bottom of Ψ are the subwords of the hub $W_M^{\pm 1}$, i.e., the history H of Ψ and H^{-1} are the histories of M_4 -accepting subtrapezia of Ψ . Therefore, by Property (xii) (b), the history H is of type (3). We may assume that the base of Ψ has a subword $(k't')^\epsilon$ with $\epsilon = 1$ since otherwise one can replace Ψ by its mirror copy. Let Γ be the maximal subtrapezium of Ψ with base $k't'$. Then every cell of the maximal k' -band \mathcal{C} of Γ is active from the right by Property (ii). But the a -bands starting on \mathcal{C} cannot end on the passive (see Property (ii)) t' -band of Γ . They also cannot end on \mathcal{C} by Property (vii). Hence $\|H\| = 0$, a contradiction.

Thus, any two hubs of a reduced diagram cannot be connected by two t -bands, such that the subdiagram bounded by them contains no other hubs. This property makes the hub graph of a reduced diagram hyperbolic, in a sense, since the degree L of every proper vertex (=hub) is high ($L \geq 40$). Below we give a more precise formulation (proved for diagrams with such a hub graph, in particular, in [19], Lemma 11.4 and in [11], Lemma 3.2).



Lemma 5.18. *If a reduced diagram over the group G contains a least one hub, then there is a hub Π in Δ such that $L - 3$ consecutive maximal t -bands $\mathcal{B}_1, \dots, \mathcal{B}_{L-3}$ start on $\partial\Delta$, end on the boundary $\partial\Pi$, and for any $i \in [1, L - 4]$, there are no discs in the subdiagram Γ_i bounded by $\mathcal{B}_i, \mathcal{B}_{i+1}, \partial\Pi$, and $\partial\Delta$.*

A maximal q -band starting on a hub of a diagram is called a *spoke*.

Lemma 5.18 implies

Lemma 5.19. *If a reduced diagram Δ has $m \geq 1$ hubs then the number of q -edges in the boundary path of Δ is greater than $mLN/2$.*

Proof. Δ has a hub Π satisfying the assumption of Lemma 5.18. Then we can separate a subdiagram with only one hub Π from Δ by making cuts along the t -bands $\mathcal{B}_1, \mathcal{B}_{L-3}$, and along the part of $\partial\Pi$ having 3 t -edges. Since by Lemma 5.6, every spoke of Γ_i ($i \in [1, L - 4]$) starting on Π must end on $\partial\Pi$, the remaining diagram Δ' with $m - 1$ hubs has at most $|\partial\Delta|_q - (L - 4)N + 4N$ q -edges in the boundary. Since $L - 8 > L/2$ the statement follows by induction on m . \square

5.4 Parameters

The following constants will be used for the proofs in this paper.

$$L, N \ll J \ll \delta^{-1} \ll (\delta')^{-1} \ll c_0 \ll c_1 \ll \cdots \ll c_7 \quad (5.9)$$

For each of the inequalities of this paper, one can find the highest constant (with respect to the order \ll) involved in the inequality and see that for fixed lower constants, the inequality is correct as soon as the value of the highest one is sufficiently large. This principle makes the system of all inequalities used in this paper consistent.

5.5 Modified length of words and paths.

Recall that the standard length $||w||$ of a word (a path) is called the *combinatorial length*. To introduce new length function on the group words in the generators of the groups M and G we first consider a word w having no q -letters. We set the length $|a|$ of every a -letter a equal to δ . We set the length of any θ -letter equal to 1, but the the length $|v|$ of any θa -syllable, i.e., a 2-letter word v with one θ -letter and one a -letter, will be equal to $1 + \delta'$. The *length of a decomposition* of w in a product of letters and θa -syllables is the sum of lengths of the factors of this decomposition. The *length* $|w|$ of w is the smallest length of such decompositions. Finally, the length $|W|$ of arbitrary word $W \equiv w_0 u_1 \dots u_n w_n$, where u_i -s are q -letters and the words w_j -s have no q -letters, is, by definition, $n + \sum_{i=0}^n |w_i|$. The *length of a path* in a diagram is the length of its label. The *perimeter* $|\partial\Delta|$ of a diagram is similarly defined by cyclic decompositions of its boundary $\partial\Delta$.

Why do we need such a modification ? The assumption that a -edges are much shorter than other edges is used in Lemma 13.2 (Step (2)) and in other lemmas. The assumption that $\delta' \ll \delta$, and so the length of a θa -syllable is less than the sum of lengths of its letters, is used in Lemma 7.18 and in many other lemmas.

Lemma 5.20. *Let \mathbf{s} be a path in a diagram Δ , having d a -edges and e non- a -edges. Then*

- (a) $e + d\delta \geq |\mathbf{s}| \geq e + d\delta' + \max(0, (d - e)(\delta - \delta')) \geq e + d\delta'$;
- (b) *if $\mathbf{s} = \mathbf{s}_1 \mathbf{s}_2$, then $|\mathbf{s}_1| + |\mathbf{s}_2| \geq |\mathbf{s}| \geq |\mathbf{s}_1| + |\mathbf{s}_2| - (\delta - \delta')$ and $|\mathbf{s}| = |\mathbf{s}_1| + |\mathbf{s}_2|$ if \mathbf{s}_1 ends or \mathbf{s}_2 starts with a q -edge or if both these edges are not a -edges;*
- (c) *if \mathbf{s} is a top or a bottom of a q -band having h cells, then $h \leq |\mathbf{s}| \leq h(1 + \delta')$; and $|\mathbf{s}| = h$ if \mathbf{s} has no a -edges.*
- (d) $||\mathbf{s}|| \geq |\mathbf{s}| \geq \delta ||\mathbf{s}||$.

Proof. (a) Since every path is a product of q -, θ -, and a -edges, the first inequality follows. The second one is true because at most e a -edges can be joined with θ -letters to form 2-edge subpaths of \mathbf{s} , and the remaining a -edges has to be taken alone with coefficient δ when one calculate $|\mathbf{s}|$. To make the reader more familiar with the definition of $|\mathbf{s}|$, we leave claims (b), (c), (d) for exercises. \square

6 Mixture on the boundary of a diagram

Let O be a circle with two-colored finite set of points (or vertices) on it, more precisely, let any vertex of this finite set be either black or white. We call O a *necklace* with black and white *beads* on it. We want to introduce the *mixture* of this finite set of beads.

Assume that there are n white beads and n' black ones on O . We define sets \mathbf{P}_j of ordered pairs of distinct white beads as follows. A pair (o_1, o_2) ($o_1 \neq o_2$) belongs to the set \mathbf{P}_j if the simple arc of O drawn from o_1 to o_2 in clockwise direction has at least j black beads. We denote by $\mu_J(O)$ the sum $\sum_{j=1}^J \text{card} \mathbf{P}_j$ (the J -mixture on O). Below similar sets for another necklace O' are denoted by \mathbf{P}'_J . In this section, $J \geq 1$, but later on it will be a fixed large enough number J from the list (5.9).

Lemma 6.1. (a) $\mu_J(O) \leq J(n^2 - n)$.

(b) Suppose a necklace O' is obtained from O after removal of a white bead v . Then $\text{card} \mathbf{P}_j - n < \text{card} \mathbf{P}'_j \leq \text{card} \mathbf{P}_j$ for every j , and $\mu_J(O) - Jn < \mu_J(O') \leq \mu_J(O)$.

(c) Suppose a necklace O' is obtained from O after removal of a black bead v . Then $\text{card} \mathbf{P}'_j \leq \text{card} \mathbf{P}_j$ for every j , and $\mu_J(O') \leq \mu_J(O)$.

(d) Assume that there are three beads v_1, v_2, v_3 of a necklace O , such that the clockwise arc $v_1 - v_3$ contains v_2 and has at most J black beads (excluding v_1 and v_3), and the arcs $v_1 - v_2$ and $v_2 - v_3$ have m_1 and m_2 white beads, respectively. If O' is obtained from O by removal of v_2 , then $\mu_J(O') \leq \mu_J(O) - m_1 m_2$.

Proof. (a) It is clear from the definition that $\text{card} \mathbf{P}_j \leq n^2 - n$, and the statement (a) follows. The statements (b) and (c) are obvious.

(d) Let o (o') be a white bead on $v_1 - v_2$ (on $v_2 - v_3$). Then for some $j \in \{1, \dots, J\}$, the pair (o, o') belongs to \mathbf{P}_j but does not belong to \mathbf{P}_{j+1} . Now, on the one hand, the same pair (o, o') considered on O' does not belong to \mathbf{P}'_j . On the other hand, we clearly have $\mathbf{P}'_j \subseteq \mathbf{P}_j$. Therefore $\mu_J(O) - \mu_J(O')$ is at least the number of such pairs (o, o') , which is equal to $m_1 m_2$. The lemma is proved. \square

We will use also the mixture of beads on a closed interval $\mathbf{x} = [a, b]$ with real $a < b$. A *string of beads* is a finite sets of white and black beads on \mathbf{x} , but in the definition of mixture $\mu^c(\mathbf{x})$ we consider only pairs (o, o') of white beads, where $o < o'$. This gives us the mixture $\mu_J^c(\mathbf{x})$ as above.

Lemma 6.2. Let \mathbf{x} be a string of beads and $J \geq 1$.

(a) $\mu_J^c(\mathbf{x}) \leq J(n^2 - n)/2$.

(b) Suppose a string \mathbf{x}' is obtained from \mathbf{x} after removal of a white bead v . Then $\text{card} \mathbf{P}_j - n < \text{card} \mathbf{P}'_j \leq \text{card} \mathbf{P}_j$ for every j , and $\mu_J^c(\mathbf{x}) - Jn < \mu_J^c(\mathbf{x}') \leq \mu_J^c(\mathbf{x})$.

(c) Suppose a string \mathbf{x}' is obtained from \mathbf{x} after removal of a black bead v . Then $\text{card} \mathbf{P}'_j \leq \text{card} \mathbf{P}_j$ for every j , and $\mu_J^c(\mathbf{x}') \leq \mu_J^c(\mathbf{x})$.

(d) Assume that there are three black beads $v_1 < v_2 < v_3$ on \mathbf{x} such that the interval (v_1, v_3) has at most J black beads, and the intervals (v_1, v_2) and (v_2, v_3) have m_1 and m_2 white beads, respectively. If \mathbf{x}' is obtained from \mathbf{x} after removal of the bead v_2 , then $\mu_J^c(\mathbf{x}') \leq \mu_J^c(\mathbf{x}) - m_1 m_2$.

(e) Assume that the set of black beads is non-empty. Then there is a black bead v , such that it divides \mathbf{x} into two subsegments with m_1 and m_2 white beads, respectively, $m_1 \geq m_2$, and $m_1 m_2 \leq \mu_1^c(\mathbf{x}) \leq (2m_1 - 1)m_2$.

Proof. The proof of statements (a) - (d) is similar to the proof of Lemma 6.1. To prove claim (e), we choose the black bead v so that the difference $|m_1 - m_2|$ is minimal. We can assume that $m_1 \geq m_2$. Since m_1 white beads are separated by v from m_2 white beads,

we have $\mu_1^c(x) \geq m_1 m_2$. On the other hand, there is a subsegment with $m_1 - m_2$ pairwise non-separated (by black beads) white beads. Therefore

$$\mu_1^c(x) \leq \frac{1}{2}(m_1 + m_2)(m_1 + m_2 - 1) - \frac{1}{2}(m_1 - m_2)(m_1 - m_2 - 1) = (2m_1 - 1)m_2$$

□

For any diagram Δ , we introduce the following invariant $\kappa(\Delta) = \mu_1(\partial\Delta)$. To define them, we consider the boundary $\partial(\Delta)$, as a κ -necklace, i.e., we consider a circle O with $||\partial\Delta||$ edges labeled as the boundary path of Δ . By definition, the white beads are the mid-points of the θ -edges of O and black beads are the mid-points of the q -edges O . Then, by definition, the κ -mixture on $\partial\Delta$ is $\kappa(\Delta) = \mu_1(O)$.

We will need an analogous parameter $\nu_J(\Delta)$. The definition of the ν -necklace on $\partial\Delta$ is similar, but the black beads of it correspond to t - and t' -edges only while the set of white beads coincides with that for the κ -necklace. The ν -necklace has ν_J -mixture for every $J \geq 1$, which is called the ν_J -mixture on $\partial\Delta$ and denoted by $\nu_J(\Delta)$.

Recall that a θ -letter is said to be (12)-letter ((23)-letter) if it corresponds to the rule (12) (to (23)). Such a letter is *special* if it is involved in a ((12), t)-relation or in a ((23), t')-relation. An edge is a (12)-edge (a (23)-edge, a *special* edge) if it is labeled by a (12)-letter (by a (23)-letter, by a special θ -letter, respectively). Note that if a q -band \mathcal{C} has a special (12)-edge (a special (23)-edge) on the left side, then the base of \mathcal{C} is either a $t^{\pm 1}$ or a k (resp., either a $(t')^{\pm 1}$ or a $(k')^{-1}$).

To define an auxiliary parameter $\lambda(\Delta)$ we consider, the λ -necklace, where white beads are the middle points of all θ -edges of O which are neither (12)-edges nor (23)-edges, and the black beads are the middle points of all non-special (12)- and (23)-edges and all q -edges of O . The λ -necklace defines the λ_J -mixture on $\partial\Delta$ for every J , and for $J = 1$, we denote it by $\lambda(\Delta)$.

By definition, $\mu(\Delta) = c_0 \kappa(\Delta) + \lambda(\Delta)$. The ν_J -mixtures on the boundaries will be later applied for a large enough J .

Similarly we have $\kappa(\mathbf{x})$, $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$, and $\nu_J(\mathbf{x})$ for any path \mathbf{x} in a diagram. (Consider the strings of beads to define.) Clearly, each of this values remains unchanged if one replaces \mathbf{x} by \mathbf{x}^{-1} .

7 General properties of combs

By Lemma 5.10, every property of a trapezium can be formulated as a property of a computation of the S-machine M , and vice versa. Unfortunately minimal diagrams can be much more complicated than trapezia. Now we define diagrams which are the main subject of our research in this paper.

As in [15], we say that a reduced diagram Γ over M with reduced boundary path (having no subpaths of the form ee^{-1}) is a *comb* if it has a rim q -band \mathcal{C} (the *handle* of the comb), and every maximal θ -band of Γ has a cell in \mathcal{C} . In particular, every trapezium is a comb.

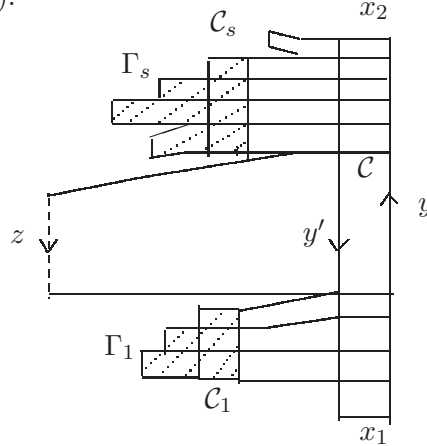
Suppose that a maximal q -band \mathcal{C} of a diagram Δ starts and ends on $\partial\Delta$. Then it divides Δ into two subdiagrams Γ and Γ' , where Γ' contains \mathcal{C} . Suppose Γ is a comb with handle \mathcal{C} . Then we call Γ a *subcomb* of Δ .

By Lemma 5.6, any maximal q -band \mathcal{C}' of a comb Γ is itself a handle of a subcomb Γ' of Γ which does not contain (by definition of subcomb of a comb) cells from the handle \mathcal{C} of Γ if $\mathcal{C}' \neq \mathcal{C}$. In this case Γ' is a *proper* subcomb of the comb Γ .

The *base width* of a comb is by definition the maximal number of letters in the bases of its θ -bands. The *history* H and the Step history of a comb are the history and Step history of its handle. If H' is a subword of H then H' -part of the comb is the union of all maximal θ -bands corresponding to H' .

It will be convenient to view a comb Γ with the handle on its right. Thus the bottom of the handle \mathcal{C} is the right side of \mathcal{C} , and it is the part of $\partial\Gamma$. Respectively, every q -band of Γ has the right side and the left side. The words written on tops/bottoms of θ -bands of Γ and their bases will be read from left to right, and so, for a base letter q , one can distinguish q - and q^{-1} -bands of Γ . In particular, a q -band of Γ can be active from the left, active from the right (or passive). If a q -band \mathcal{D} is passive from the left (from the right), then $h = |\mathbf{y}'| \leq |\mathbf{y}|$, where h is the number of cells in \mathcal{D} , (resp., $|\mathbf{y}'| \geq |\mathbf{y}| = h$) by the definition of length and Lemma 5.20.

We introduce the following permanent notation for a comb Γ with a handle \mathcal{C} . Denote by H the history of \mathcal{C} and set $h = ||H||$, i.e., h is the length of \mathcal{C} , the number of q -cells in \mathcal{C} . The comb Γ is a *one-Step* comb if the history H is one-Step, i.e., H has one of the types (1), (2), or (3).



The boundary of \mathcal{C} is $\mathbf{x}_1\mathbf{y}\mathbf{x}_2\mathbf{y}'$, where \mathbf{x}_1 and \mathbf{x}_2 are the boundary q -edges of the band \mathcal{C} and $\mathbf{y}\mathbf{z}$ is the boundary of Γ . (Thus, \mathbf{y} is the right side of \mathcal{C} , and $(\mathbf{y}')^{-1}$ is the left side.) Similarly we have the decomposition $(\mathbf{y}')^{-1}\mathbf{z}'$ for the boundary of $\Gamma \setminus \mathcal{C}$, where $\mathbf{z} = \mathbf{x}_2\mathbf{z}'\mathbf{x}_1$. By definition, $\text{Area}'(\Gamma) = \text{Area}(\Gamma \setminus \mathcal{C})$. Since \mathbf{z} starts (ends) with the q -edge \mathbf{x}_2 (with \mathbf{x}_1), we have $|\partial\Gamma| = |\mathbf{y}| + |\mathbf{z}|$ by Lemma 5.20 (b). We also use $\mathbf{y}^\Delta, \mathbf{z}^\Delta, \dots$ instead of $\mathbf{y}, \mathbf{z}, \dots$ if we want to stress that the notation relates to a particular comb Δ .

Remark 7.1. It follows from Lemma 5.6 that every maximal θ -band crossing the handle of a comb Δ must ends on \mathbf{z}^Δ . Therefore $|\mathbf{y}^\Delta|_\theta = |\mathbf{y}'^\Delta|_\theta = |\mathbf{z}^\Delta|_\theta = h$.

For a comb Γ , we modify the notion of mixture. The *comb mixtures* are $\kappa^c(\Gamma) = \kappa(\mathbf{z}) - \kappa(\mathbf{y})$, $\lambda^c(\Gamma) = \lambda(\mathbf{z}) - \lambda(\mathbf{y})$, and similarly, $\nu_J^c(\Gamma) = \nu_J(\mathbf{z}) - \nu_J(\mathbf{y})$ ($\lambda(\Gamma)$ can be negative if (12)- or (23)-cells separate other θ -cells of the handle !). By definition $\mu^c(\Gamma) = c_0\kappa^c(\Gamma) + \lambda^c(\Gamma)$.

Lemma 7.2. *In the above notation, we have (a) $\kappa^c(\Gamma) \geq 0$, (b) $\nu_J^c(\Gamma) = \nu_J(\mathbf{z}) \geq 0$, (c) $\lambda^c(\Gamma) \geq 0$ if for every special edge e of \mathbf{z} , the edge f of \mathbf{y} connected with e by a θ -band, is also special.*

Proof. (a), (b) Since the path \mathbf{y} has no q -edges, we have $\kappa(\mathbf{y}) = 0$ ($\nu_J(\mathbf{y}) = 0$, respectively), and so $\kappa^c(\Gamma) = \kappa(\mathbf{z}) \geq 0$ ($\nu_J^c(\Gamma) = \nu_J(\mathbf{z}) \geq 0$, resp.).

(c) Consider the strings of beads on \mathbf{z} and on \mathbf{y} used in the definitions of the comb mixture $\lambda^c(\cdot)$. By Lemma 5.6, the maximal θ -bands of Γ establish a bijection between the white vertices of \mathbf{z} and white vertices of \mathbf{y} , preserving the order of the beads on \mathbf{z} and \mathbf{y}^{-1} , respectively. Every black bead on \mathbf{y} must belong to a non-special θ -edge f . By the condition of the lemma, we have a black bead on the corresponding edge e of \mathbf{z} . Hence one can apply Lemma 6.1 (c) to the strings of beads on \mathbf{z} and \mathbf{y} several times to conclude that $\lambda(\mathbf{z}) \geq \lambda(\mathbf{y})$, and so $\lambda^c(\Gamma) \geq 0$. \square

Lemma 7.3. *Let Γ be a proper subcomb of a diagram (of a comb) Δ . Let $\Delta \setminus \Gamma$ be the compliment of Γ in Δ , whose handle is the handle of Δ if Δ is a comb. Then*

- (a) $\kappa(\Delta \setminus \Gamma) \leq \kappa(\Delta) - \kappa^c(\Gamma)$ and $\lambda(\Delta \setminus \Gamma) \leq \lambda(\Delta) - \lambda^c(\Gamma)$,
- (b) $\nu_J(\Delta \setminus \Gamma) \leq \nu_J(\Delta) - \nu_J^c(\Gamma)$ for every $J \geq 1$,
- (c) $\kappa^c(\Delta \setminus \Gamma) \leq \kappa^c(\Delta) - \kappa^c(\Gamma)$ and $\lambda^c(\Delta \setminus \Gamma) \leq \lambda^c(\Delta) - \lambda^c(\Gamma)$ if Δ is a comb,
- (d) $\nu_J^c(\Delta \setminus \Gamma) \leq \nu_J^c(\Delta) - \nu_J^c(\Gamma)$ for every $J \geq 1$ if Δ is a comb,
- (e) If $\bar{\Delta}$ is a subcomb of a diagram Δ and Γ is a subcomb of $\bar{\Delta}$, then for every $J \geq 1$, $0 \leq \nu_J^c(\bar{\Delta}) - \nu_J^c(\bar{\Delta} \setminus \Gamma) \leq \nu_J(\Delta) - \nu_J(\Delta \setminus \Gamma)$. (Also we have $\nu_J^c(\bar{\Delta}) - \nu_J^c(\bar{\Delta} \setminus \Gamma) \leq \nu_J^c(\Delta) - \nu_J^c(\Delta \setminus \Gamma)$ if Δ is a comb.)

Proof. (a) Let $\mathbf{y} = \mathbf{y}^\Gamma$, and $\mathbf{z} = \mathbf{z}^\Gamma$, \mathbf{zx} the boundary path of Δ . To obtain the necklace O' corresponding to $\Delta \setminus \Gamma$, one replaces the subpath \mathbf{z} of the boundary by \mathbf{y}^{-1} . Therefore the pairs of white beads counted to get $\lambda(\mathbf{z})$ are replaced by pairs counted to get $\lambda(\mathbf{y})$. (Note that the white beads of \mathbf{z} are in bijective correspondence with white beads of \mathbf{y} by the definition of comb and Lemma 5.6.) Since every white bead of \mathbf{z} is separated from any white bead of $\partial\Delta \setminus \partial\Gamma$ by the black beads in the middle of the first and the last edges of \mathbf{z} , Inequality (a) is proved for λ -mixtures. The case of κ -mixtures is similar.

The proofs of claims (b), (c), and (d) are also similar.

The path \mathbf{y}^Γ has no t -edges, and the first inequality of (e) follows. Similarly, every pairs of white beads which makes a contribution to $\nu_J^c(\bar{\Delta})$ but not to $\nu_J^c(\bar{\Delta} \setminus \Gamma)$ also contributes to $\nu_J(\Delta)$ but not to $\nu_J(\Delta \setminus \Gamma)$, and the second inequality of (e) follows. The proof of the version in the parentheses is similar. \square

Let Γ be a comb and $\mathbf{z}^1, \dots, \mathbf{z}^r$ the maximal subpaths of $\mathbf{z} = \mathbf{z}^\Gamma$ containing no q -edges. We denote by l^1, \dots, l^r their θ -lengths, and define $l_- = l_-^\Gamma$ to be $h - \max_{i=1}^r l^i$. (Note that $h = h^\Gamma = \sum l^i$ by Lemma 5.6.)

A θ -band which starts on the handle \mathcal{C} of a comb Γ will be called *simple* if it has no (θ, q) -cells except for the cell of \mathcal{C} , and is maximal with respect to this property.

We call a maximal q -band \mathcal{B} a *derivative* band, if it is not \mathcal{C} but it can be connected with \mathcal{C} by a simple θ -band. Throughout the paper, we will use notation $\mathcal{C}_1, \dots, \mathcal{C}_s$ for derivative bands of a comb Γ . It is possible that $s = 0$, and every maximal θ -band is simple in this case.

Every derivative band \mathcal{C}_i is a handle of a subcomb Γ_i (which does not contain \mathcal{C}). We will use this notation and call Γ_i a *derivative subcomb* of Γ . It follows from the definitions

that every cell of a comb belongs either to a derivative subcomb or to a simple band of Γ .

Recall that every maximal θ -band of a comb, in particular, a maximal θ -band crossing a derivative band \mathcal{C}_i , must cross the handle \mathcal{C} . Therefore every cell of \mathcal{C}_i is connected with \mathcal{C} by a θ -band. Since there is a simple θ -band among these θ -bands, no other derivative \mathcal{C}_j can intersect these connecting θ -bands by Lemma 5.6, i.e., all of them are simple. It follows that different derivative subcombs are disjoint, and if $\mathcal{C}_1, \dots, \mathcal{C}_s$ is the system of all derivative bands in Γ with histories H_1, \dots, H_s , then H_1, \dots, H_s are pairwise disjoint subwords in the history H of Γ . Therefore $\sum_{i=1}^s h_i \leq h$, where $h_i = \|H_i\|$. We will also use h_- for $\sum_{i=1}^s h_i - \max_{i=1}^s h_i$.

Lemma 7.4. *In the above notation, $l_- \geq \min(\sum_{i=1}^s h_i, h - \max_{i=1}^s h_i)$. In particular,*

$$h_- \leq l_- \quad (7.10)$$

Proof. Let $|\mathbf{z}^{i_0}|_\theta = l^{i_0} = \max_{i=1}^r l^i$. Then, either every maximal θ -band ending on \mathbf{z}^{i_0} crosses some derivative band \mathcal{C}_j , where $j = j(i_0)$, or every maximal θ -band crossing \mathbf{z}^{i_0} crosses no derivative bands because otherwise a q -band would cross \mathbf{z}^{i_0} . (This follows from the definitions of comb, of \mathbf{z}^i -s and from Lemma 5.6.) In the former case, $l_- = h - l^{i_0} \geq h - \max_{i=1}^s h_i \geq h_-$, and in the latter case, $l_- = h - l^{i_0} \geq \sum_{j=1}^s h_j \geq h_-$. \square

Lemma 7.5. *In the above notation, we have $hh_- \leq hl_- \leq 2\kappa^c(\Gamma)$.*

Proof. By (7.10), it suffices to prove the second inequality. There are h white beads on \mathbf{z} . Every such a bead o belongs to one of the paths \mathbf{z}^i having θ -length at most $\max_{i=1}^r l^r$. Therefore for every such o , there are at least l_- white beads o' on \mathbf{z} such that o and o' are separated on \mathbf{z} by a black bead. Thus, we obtain at least hl_- pairs (o, o') of white beads on \mathbf{z} separated by black beads. Since one of the pairs (o, o') and (o', o) contributes 1 to $\kappa^c(\Gamma)$, the lemma is proved. \square

Let the handle \mathcal{C} of a comb Γ is a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band with history having no (23)-rules or no (12)-rules, respectively; and every derivative \mathcal{C}_i is a $k^{\pm 1}$ - or a $(k')^{\pm 1}$ -band such that there are no special θ -edges (corresponding to the rules (12) and (23)) in the derivative subcomb Γ_i . A subband \mathcal{B} of some \mathcal{C}_i which has neither (12)- nor (23)-edges and is maximal with respect to this property, is called a *short derivative* of \mathcal{C} . By Property (vii), there are no maximal a -bands starting and ending on the same short derivative band. Let h'_1, \dots be the lengths of all short derivatives. Let h' be the number of maximal θ -bands in Γ , which do not correspond to the rules (12) and (23). Define $h'_- = h' - \max h'_j$.

Lemma 7.6. *In the above notation, we have $hh'_- \leq 6\lambda(z^\Gamma) = 6\lambda^c(\Gamma)$.*

Proof. The sets of ends of the θ -bands crossing two short derivatives are separated in \mathbf{z}^Γ either by a q -edge or by a non-special θ -edge. Therefore arguing as in the proof of Lemma 7.5, we come to inequality $h'h'_- \leq 2\lambda(\mathbf{z}^\Gamma) = 2\lambda^c(\Gamma)$. (We note that under the assumption on the history, $\lambda(\mathbf{y}^\Gamma) = 0$ since \mathcal{C} is a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band.) This implies the statement of the lemma if $h'_- \neq 0$ since in this case we have $h' \geq h/3$ because the handle \mathcal{C} , being a reduced diagram, cannot have two consecutive cells corresponding to the rules (12), (23) (or inverse). If $h'_- = 0$ the claim of the lemma is obvious. \square

Lemma 7.7. *Let Γ be a comb with a handle \mathcal{C} of length h . Then the number a_{ij} of all maximal a -bands of Γ starting on a derivative band \mathcal{C}_i and ending on the bands \mathcal{C}_j with $j \neq i$ is at most h_- . The total number of cells in these a -bands over all $i < j$ does not exceed $hh_- \leq 2\kappa^c(\Gamma)$.*

Proof. Recall that derivative subcombs with different handles \mathcal{C}_i and \mathcal{C}_j are disjoint and separated by these handles (which are q -bands) from the remaining part of Γ . Therefore every a -band \mathcal{A} connecting some \mathcal{C}_i and \mathcal{C}_j ($i \neq j$), connects a -edges of cells on the right sides of these derivative bands. But every q -cell of \mathcal{C}_i has at most one a -edge on the right side of it by (iii) (b). Besides, a connecting a -band \mathcal{A} under consideration either starts or ends on some \mathcal{C}_j , where $j \neq i$. Thus the total number of all connecting a -bands cannot exceed $h - h_r$ for arbitrary $r \leq s$. Now the first statement of the lemma follows from the definition of h_- . Since the number of cells in \mathcal{A} is at most h by Lemma 5.6, the second statement is also proved by Lemma 7.5. \square

Lemma 7.8. *Let Γ be a comb and let its handle \mathcal{C} be a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band with history having no (23)-rules or (12)-rules, respectively, and every derivative \mathcal{C}_i is a $k^{\pm 1}$ - or a $(k')^{\pm 1}$ -band such that there are no special θ -edges in the derivative subcomb Γ_i . Let \mathcal{B}_1, \dots be the system of all short derivative bands. Then the number of all the maximal a -bands of Γ starting on a short derivative \mathcal{B}_i and ending on some \mathcal{B}_j ($j \neq i$), where \mathcal{B}_i and \mathcal{B}_j are subbands of the same derivative band, is at most h'_- . The total number of cells in these a -bands over all i, j does not exceed $hh'_- \leq 6\lambda(\mathbf{z}^\Gamma) \leq 6\lambda^c(\Gamma)$.*

Proof. The proof is similar to the proof of Lemma 7.7, but one should use Lemma 7.6 instead of Lemma 7.5. \square

Lemma 7.9. (a) *Let $\Gamma_1, \dots, \Gamma_s$ be the derivative subcombs of a comb Γ . Then $\sum_{i=1}^s \kappa^c(\Gamma_i) \leq \kappa^c(\Gamma)$.*

(b) *If the history of a comb Γ is $H \equiv H(1) \dots H(t)$, and $\Gamma(1), \dots, \Gamma(t)$ are $H(1)$ -, \dots , $H(t)$ -parts of Γ , resp. ($\Gamma(i)$ is absent if H_i is empty), then $\sum_{i=1}^t \kappa^c(\Gamma(i)) \leq \kappa^c(\Gamma)$.*

Proof. (a),(b) Note that every white bead of Γ_i (of $\Gamma(i)$) is placed on the boundary of Γ , and two white beads of $\partial\Gamma_i$ separated by a black bead are also separated by the same black bead on $\partial\Gamma$. Since the sets of white beads of Γ_i and Γ_j (of $\Gamma(i)$ and $\Gamma(j)$) are disjoint for $i \neq j$, the statements (a) and (b) follow from the definition of $\kappa^c(\cdot)$. \square

Lemma 7.10. (a) *Let Γ be a comb. Then the number α of a -edges in $\mathbf{z}' = (\mathbf{z}')^\Gamma$ does not exceed $(\delta')^{-1}(|\mathbf{z}'| - h) = (\delta')^{-1}(|\mathbf{z}| - h - 2)$.*

(b) *Assume in addition that the handle \mathcal{C} is passive from the left and there are no derivative bands \mathcal{C}_i such that some non-trivial a -band starts and ends on $\partial\mathcal{C}_i$. Then the total area of all simple θ -bands $\mathcal{S}_1, \dots, \mathcal{S}_h$ of Γ is at most*

$$h(h_- + \alpha + 1) \leq h(h_- + (\delta')^{-1}(|\mathbf{z}| - h - 1)).$$

Proof. (a) Notice that $|\mathbf{z}'| \geq h + \delta'\alpha$ by Lemmas 5.6 and 5.20 (a), and so $\alpha \leq (\delta')^{-1}(|\mathbf{z}'| - h) = (\delta')^{-1}(|\mathbf{z}| - h - 2)$.

(b) The total number of cells in all a -bands connecting the derivative bands is at most hh_- by Lemma 7.7. If a (θ, a) -cell π of a simple θ -band \mathcal{S}_j does not belong to any such connecting a -bands, then one of the ends of the maximal a -band containing π must belong to $\partial\Gamma$ because \mathcal{C} is passive from the left. The number of such a -bands is at most

α , and the total number of their cells is at most αh , because their lengths do not exceed h by Lemma 5.6. Since a simple band has one q -cell, the number of cells in all the simple θ -bands is at most $h(h_- + \alpha + 1)$. \square

Remark 7.11. If \mathcal{C}_i is a derivative band of a comb Γ , then every a -band connecting two cells from \mathcal{C}_i is of length at most h_i , and the total area of such bands crossing simple bands of Γ at most $h_i^2/2$. Hence if we omit the assumption that there are no derivative bands \mathcal{C}_i such that some non-trivial a -band starts and ends on \mathcal{C}_i , then we may add $\sum_{i=1}^s h_i^2/2 \leq \frac{h}{2} \sum_{i=1}^s h_i$ to the estimate of Lemma 7.10 (b), and in this case the total number cells n_s in all simple bands satisfies

$$n_s \leq h(h_- + \alpha + 1) + \sum_{i=1}^s h_i/2 \leq h((\delta')^{-1}(|\mathbf{z}| - h - 1) + \frac{3}{2} \sum_{i=1}^s h_i) \quad (7.11)$$

If we use both Lemmas 7.7 and 7.8 instead of Lemma 7.7 in the proof of Lemma 7.10 we get

Lemma 7.12. *Let Γ be a comb and let its handle \mathcal{C} be a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band with history having no (23)-rules or no (12)-rules, respectively, and every derivative \mathcal{C}_i is a $k^{\pm 1}$ - or a $(k')^{\pm 1}$ -band such that there are no special θ -edges in the derivative subcombs of Γ_i . Then the total area of all simple θ -bands of Γ is at most $h(h_- + h'_- + \alpha + 1) \leq h(h_- + h'_- + (\delta')^{-1}(|\mathbf{z}| - h - 1))$, where α is the number of a -edges in $\mathbf{z}' = (\mathbf{z}')^\Gamma$*

\square

The proof of the following lemma can be obtained from the proof of [15, Lemma 4.10] by replacing $|\Gamma|_a$ by $|\mathbf{z}|_a$ and replacing the constant C by 2 (since C was the maximum of the numbers of a -letters in (θ, q) -relations in [15]).

Lemma 7.13. *Let h and b be the height and the base width of a comb Γ , respectively, and let $\mathcal{T}_1, \dots, \mathcal{T}_h$ be consecutive θ -bands of Γ . We can assume that $\mathbf{bot}(\mathcal{T}_1)$ and $\mathbf{top}(\mathcal{T}_h)$ are contained in $\partial\Gamma$. Let $\alpha = |\mathbf{z}^\Gamma|_a$, and we denote by α_1 the number of a -edges on $\mathbf{bot}(\mathcal{T}_1)$. Then $\alpha + 8hb \geq 2\alpha_1$, and the area of Γ does not exceed $4bh^2 + 2\alpha h$.*

Remark 7.14. For a comb Γ , we will use symbol $[\Gamma]$ to denote the product $h^\Gamma(|\mathbf{z}^\Gamma| - |\mathbf{y}^\Gamma|)$. As we noted in Introduction, an estimate of the form $\text{Area}(\Gamma) \leq C[\Gamma]$ (where C does not depend on Γ) would be perfect for the proof of the main theorem. It follows from the definition of comb that every maximal θ -band of Γ starting on \mathbf{y}^Γ ends on \mathbf{z}^Γ and vice versa, that is $|\mathbf{z}^\Gamma|_\theta = |\mathbf{y}^\Gamma|_\theta = h^\Gamma$. Clearly $|\mathbf{z}^\Gamma|_q - 2 \geq |\mathbf{y}^\Gamma|_q = 0$ since the path \mathbf{z}^Γ contains at least 2 q -edges of the handle of Γ , and therefore, when we estimate $|\mathbf{z}^\Gamma| - |\mathbf{y}^\Gamma|$ from below in the proofs of several lemmas, our goal is to obtain a lower bound for the difference $|\mathbf{z}^\Gamma|_a - |\mathbf{y}^\Gamma|_a$.

We observed earlier that if the handle of a comb Γ is passive from the right, then $|\mathbf{y}^\Gamma| = h^\Gamma$, and so $[\Gamma]$ is equal to $h^\Gamma(|\mathbf{z}^\Gamma| - h^\Gamma)$, and therefore it is positive. Moreover:

Lemma 7.15. *If the height h of a comb Γ does not exceed $(\delta')^{-1}$, then $|\mathbf{z}^\Gamma| - |\mathbf{y}^\Gamma| > 0$ and the area of Γ does not exceed $4(\delta')^{-1}[\Gamma]$.*

Proof. By Lemma 5.20 (c), $|\mathbf{y}^\Gamma| \leq h^\Gamma + 1$. If the base width of Γ is b , then, by Lemma 5.6, \mathbf{z}^Γ has at least $2b$ q -edges and at least h^Γ θ -edges. Hence by Lemma 5.20(a), $|\mathbf{z}^\Gamma| \geq$

$2b + h + \delta'|\mathbf{z}^\Gamma|_a$. Therefore $|\mathbf{z}^\Gamma| - |\mathbf{y}^\Gamma| \geq (2b - 1) + \delta'|\mathbf{z}^\Gamma|_a > 0$ since $b \geq 1$. Then, by Lemma 7.13,

$$\text{Area}(\Gamma) \leq 4b(h^\Gamma)^2 + 2|\mathbf{z}^\Gamma|_a h^\Gamma \leq h^\Gamma (\delta')^{-1} (4(2b - 1) + 2\delta'|\mathbf{z}^\Gamma|_a) \leq 4(\delta')^{-1} h^\Gamma (|\mathbf{z}^\Gamma| - |\mathbf{y}^\Gamma|)$$

□

Remark 7.16. Further we are finding appropriate estimates for the areas of combs Γ -s or for the areas of some proper subcombs of them provided the base width b of Γ is not too small and not too large. It is not small in some lemmas because we need a choice to select a suitable subcomb of Γ , and b is not too large since the estimates of Lemma 7.13 and of other lemmas depend on b . The sufficiency of the upper bound $b \leq 15N$ will be seen later.

Lemma 7.17. *Let Γ be a comb with base width $b \leq 15N$ and with passive handle \mathcal{C} . Assume that Γ has a derivative band \mathcal{C}_{i_0} which contains an active from the right subband $\tilde{\mathcal{C}}$ of length h_0 . Assume also that at most $(1 - \delta)h_0$ maximal a -bands starting on $\tilde{\mathcal{C}}$ and ending on one of the bands $\mathcal{C}, \mathcal{C}_1, \dots, \mathcal{C}_s$. Then (a) $\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma]$ if $h_0 \geq \delta h$; (b) $\sum_{i=1}^s \text{Area}(\Gamma_i) \leq (\delta')^{-2}[\Gamma]$ for the set of derivative subcombs $\Gamma_1, \dots, \Gamma_s$ if $h_0 \geq \delta \sum h_i$.*

Proof. To prove statement (a), we consider two cases.

Case 1. Assume that $\alpha = |\mathbf{z}|_a \geq \delta^2 h/2$. Then $|\mathbf{z}| - |\mathbf{y}| \geq \delta' \alpha \geq \delta' \delta^2 h/2$ by Lemma 7.10 (a) since $|\mathbf{y}| = h$. Hence by Lemma 7.13 with $b \leq 15N$, we have

$$\text{Area}(\Gamma) \leq 60N h^2 + 2\alpha h \leq h(|\mathbf{z}| - |\mathbf{y}|)(60N \times 2(\delta')^{-1} \delta^{-2} + 2(\delta')^{-1}) \leq (\delta')^{-2}[\Gamma]$$

since $(\delta')^{-1} > 120N \delta^{-2} + 2$.

Case 2. Let $\alpha = |\mathbf{z}|_a < \delta^2 h/2$. It follows from the condition of the lemma that at least $h_0 - 2$ maximal a -bands start on $\tilde{\mathcal{C}}$ but at most $(1 - \delta)h_0$ end not on \mathbf{z} . Therefore $\alpha \geq \delta h_0 - 2 \geq \delta^2 h - 2$. The arising in this case inequality $\delta^2 h - 2 < \delta^2 h/2$ implies $h < 4\delta^{-2} < (\delta')^{-1}$ by the choice of δ' . Now Claim (a) follows from Lemma 7.15.

The proof of statement (b) is similar, but now two cases appear due to the comparison of α with $\delta^2(\sum h_i)/2$, which leads to inequality $\sum h_i < 4\delta^{-2}$ in the second case. Also one takes into account inequalities $\sum h_i \leq h$ and $\sum(|\mathbf{z}_i| - |\mathbf{y}_i|) \leq |\mathbf{z}| - |\mathbf{y}|$ in both cases. □

Lemma 7.18. *Assume that a comb Γ has no maximal q -bands except for its handle \mathcal{C} , and there are no non-trivial a -bands both starting and terminating on $\mathbf{y}' = (\mathbf{y}')^\Gamma$.*

(a) *If \mathcal{C} is active from the left or passive from the left, then*

$$\text{Area}'(\Gamma) \leq (\delta')^{-1} h(|\mathbf{z}'| - |\mathbf{y}'| + 1).$$

(b) *If \mathcal{C} is active from the left or \mathcal{C} is passive (from both sides), then*

$$\text{Area}(\Gamma) \leq (\delta')^{-1}[\Gamma].$$

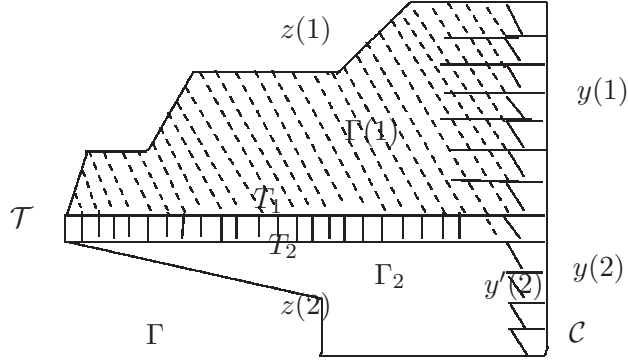
Proof. (a) Let \mathcal{T} be a longest θ -band in Γ , d the number of a -cells in \mathcal{T} . Denote by T_1 and T_2 the top and the bottom of \mathcal{T} . Consider the families \mathbf{S}_1 and \mathbf{S}_2 of a -bands starting on T_1 and T_2 , respectively, which are maximal with respect to the requirement that these bands do not contain cells from \mathcal{T} . Observe that the cardinalities $|\mathbf{S}_1|$ and $|\mathbf{S}_2|$ of these sets satisfy inequality

$$-1 \leq |\mathbf{S}_1| - |\mathbf{S}_2| \leq 1 \tag{7.12}$$

since every maximal a -band of Γ crossing T_1 has to cross T_2 (and vice versa), with at most one exception for the a -band staying on the a -edge of the unique (see (iii) (b)) (θ, q) -cell of \mathcal{T} .

If there were a non-trivial a -band from \mathbf{S}_1 and a non-trivial a -band from \mathbf{S}_2 both ending on the path \mathbf{y}' , then the maximal extension of one of them would connect different edges of \mathbf{y}' contrary to the assumption of the lemma. Therefore either no non-trivial band from \mathbf{S}_1 ends on \mathbf{y}' or no non-trivial band from \mathbf{S}_2 ends on \mathbf{y}' .

We may consider the former case only.



The path T_2 cuts Γ into two subdiagrams. We denote by $\Gamma(1)$ the subdiagram of Γ containing the bands from \mathbf{S}_1 and the θ -band \mathcal{T} . It has boundary $\mathbf{y}(1)\mathbf{z}(1)T_2$, where $\mathbf{y}(1)$ and $\mathbf{z}(1)$ are subpaths of $\mathbf{y} = \mathbf{y}(2)\mathbf{y}(1)$ and $\mathbf{z} = \mathbf{z}(1)\mathbf{z}(2)$, respectively. Similarly, $\mathbf{y}' = \mathbf{y}'(1)\mathbf{y}'(2)$, $\mathbf{z}' = \mathbf{z}'(2)\mathbf{z}'(1)$, and we define $\Gamma(2)$ as a subdiagram bounded by $\mathbf{z}(2)\mathbf{y}(2)T_2^{-1}$.

Denote by \mathbf{S} the family of maximal a -bands of $\Gamma(1)$ starting on $\partial\mathcal{C}$. It follows from the choice of $\Gamma(1)$ that the families \mathbf{S}_1 and \mathbf{S} have at most one common band starting on the intersection of \mathcal{C} and \mathcal{T} , and every a -band from these families must end on $\mathbf{z}'(1)$. Thus,

$$|\mathbf{z}'(1)|_a \geq |\mathbf{y}'(1)|_a + |\mathbf{S}_1| - 1. \quad (7.13)$$

Case 1. If \mathcal{C} is active from the left, then $|\mathbf{y}'(1)|_a \geq |\mathbf{y}'(1)|_\theta - 2$ by the definition. Note that $|\mathbf{y}'(1)|_\theta = |\mathbf{z}'(1)|_\theta$ by Lemma 5.6, and so $|\mathbf{y}'(1)|_a \geq |\mathbf{z}'(1)|_\theta - 2$. This inequality together with (7.13) imply $|\mathbf{z}'(1)|_a \geq |\mathbf{z}(1)|_\theta + |\mathbf{S}_1| - 3$, and by Lemma 5.20 (a),

$$\begin{aligned} |\mathbf{z}'(1)| &\geq |\mathbf{z}'(1)|_\theta + \delta'(|\mathbf{z}'(1)|_\theta - 3) + \delta(|\mathbf{S}_1| - 3) = |\mathbf{y}'(1)|_\theta(1 + \delta') + \delta(|\mathbf{S}_1| - 3) - 3\delta' \geq \\ &|\mathbf{y}'(1)| + \delta(|\mathbf{S}_1| - 3) - 3\delta' = |\mathbf{y}'(1)| + \delta|\mathbf{S}_1| - 3(\delta + \delta') \end{aligned} \quad (7.14)$$

Since at most $|\mathbf{S}_2|$ maximal a -bands of $\Gamma(2)$ starting on \mathcal{C} terminate on \mathcal{T} , Lemma 5.20 and (7.12) give inequality $|\mathbf{y}'(2)| \leq |\mathbf{z}'(2)| + \delta'|\mathbf{S}_2| \leq |\mathbf{z}'(2)| + \delta'(|\mathbf{S}_1| + 1)$. Therefore by (7.14),

$$|\mathbf{y}'| \leq |\mathbf{y}'(1)| + |\mathbf{y}'(2)| \leq |\mathbf{z}'(1)| + |\mathbf{z}'(2)| - \delta|\mathbf{S}_1| + 3(\delta + \delta') + \delta'(|\mathbf{S}_1| + 1).$$

Since by Lemma 5.20 (b), $|\mathbf{z}'(1)| + |\mathbf{z}'(2)| \leq |\mathbf{z}'| + \delta - \delta'$ and also $|\mathbf{S}_1| \geq d - 1$, it follows that

$$|\mathbf{y}'| < |\mathbf{z}'| - (\delta - \delta')|\mathbf{S}_1| + 4\delta \leq |\mathbf{z}'| - (d - 1)(\delta - \delta') + 4\delta,$$

whence $d \leq (\delta - \delta')^{-1}(|\mathbf{z}'| - |\mathbf{y}'| + 5\delta)$.

Case 2. If \mathcal{C} is passive from the left, we have $|\mathbf{y}'(1)| = |\mathbf{y}'(1)|_\theta$, and therefore

$$|\mathbf{z}'(1)| \geq |\mathbf{y}'(1)| + \delta'(|\mathbf{S}_1| - 1)$$

by (7.13) and by Lemma 5.20 (a). Also we have $|\mathbf{z}'(2)| \geq |\mathbf{y}'(2)|$ by Lemmas 5.6 and 5.20 (c), and so

$$|\mathbf{y}'| \leq |\mathbf{y}'(1)| + |\mathbf{y}'(2)| < |\mathbf{z}'(1)| + |\mathbf{z}'(2)| - \delta'(|\mathbf{S}_1| - 1) < |\mathbf{z}'| - \delta'(d - 1) + \delta.$$

Hence $d \leq (\delta')^{-1}(|\mathbf{z}'| - |\mathbf{y}'|) + 1 + \delta(\delta')^{-1}$.

Thus, in any case $d < (\delta')^{-1}(|\mathbf{z}'| - |\mathbf{y}'| + 1)$ because $5\delta' < \delta < 1/5$.

The number of maximal θ -bands of Γ is h , whence

$\text{Area}'(\Gamma) \leq dh < (\delta')^{-1}h(|\mathbf{z}'| - |\mathbf{y}'| + 1)$, as required.

(b) Now it follows from the assumption of the lemma and the definition of length, that $|\mathbf{y}| \leq |\mathbf{y}'| + 2\delta'$ because if \mathcal{C} is active from the left, then it has at most two passive from the left cells. However, $|\mathbf{z}| = |\mathbf{z}'| + 2$, and so $|\mathbf{z}| - |\mathbf{y}| \geq |\mathbf{z}'| - |\mathbf{y}'| + 2 - 2\delta'$. Now by (a), and inequality $3\delta' < 1$,

$$\text{Area}(\Gamma) = \text{Area}'(\Gamma) + h \leq (\delta')^{-1}h(|\mathbf{z}'| - |\mathbf{y}'| + 1 + \delta') \leq (\delta')^{-1}h(|\mathbf{z}| - |\mathbf{y}|) = (\delta')^{-1}[\Gamma]$$

□

We say that a q -band \mathcal{C}' is *close* to a q -band \mathcal{C} in a diagram Δ without hubs (i.e. over the group M) if every maximal θ -band crossing \mathcal{C}' also crosses \mathcal{C} .

Observe that a derivative band of a comb is close to the handle of this comb.

Definition 7.19. If \mathcal{C}' is close to \mathcal{C} , then there is a unique minimal subtrapezium in Δ containing both \mathcal{C}' and a subband \mathcal{B} of \mathcal{C} , where the numbers of (θ, q) -cells in \mathcal{C}' and in \mathcal{B} are equal. We will denote this *filling* subtrapezium by $Tp(\mathcal{C}', \mathcal{C})$.

Lemma 7.20. Assume that a comb Δ has no active k - or k' -cells and contains a θ -band \mathcal{T} having a subword $(pp^{-1}p)^{\pm 1}$ in the base, where p is a control letter. Then Δ has a one-Step subcomb Γ such that $\text{Area}(\Gamma) \leq (\delta')^{-1}[\Gamma]$.

Proof. Consider the maximal p -bands \mathcal{C}' and \mathcal{C}'' of Δ crossing \mathcal{T} at the (q, θ) -cells corresponding to the first and the third letters of the subword $(pp^{-1}p)^{\pm 1}$, respectively. Then \mathcal{C}' is a handle of a subcomb Δ' of Δ , and the filling trapezium $Tp(\mathcal{C}', \mathcal{C}'')$ has base $(pp^{-1}p)^{\pm 1}$. By (v), (vi), all the q -cells of $Tp(\mathcal{C}', \mathcal{C}'')$, in particular, of \mathcal{C}' , are active from both sides and has one-step history. It follows from Property (vi) applied to the comb $\Delta' \cup Tp(\mathcal{C}', \mathcal{C}'')$, that every q -band of Δ' is either active from both sides or passive, and no non-trivial a -band of Δ' can start and end on the same q -band by (vii). Hence there is a subcomb Γ of Δ' satisfying the condition of Lemma 7.18 (b), and the statement is proved. □

8 Chains and quasicombs

8.1 Intersections of chains and θ -bands

Let the boundary $\partial\pi$ of a (θ, q) -cell π have a p_i -edge for a control state letter p_i . Then by Property (iii) (a), $\partial\pi$ either has no a -edges or contains two a -edges. Below we utilize this property in the definition of chain.

Assume that $\mathcal{A}_1, \dots, \mathcal{A}_m$ are maximal a -bands such that \mathcal{A}_i terminates on an a -edge of an active p -cell π_i and \mathcal{A}_{i+1} starts with a different a -edge of π_i ($i = 1, \dots, m-1$). Then we say that $\mathcal{A}_1, \pi_1, \mathcal{A}_2, \pi_2, \dots, \mathcal{A}_m$ form a chain with $m-1$ links π_1, \dots, π_{m-1} . By Property (iii), all links are p -cells for the same control base letter $p = p_j$. A chain \mathbf{A} is called a *chain-annulus* if the first a -edge of \mathcal{A}_1 coincides with the last one of \mathcal{A}_m .

It also follows that if \mathcal{A}_1 starts with an edge e_1 and \mathcal{A}_m ends with f_m , then the letters $\text{Lab}(f_m)$ and $\text{Lab}(e_1)$ are either equal or they are copies of each other in different subalphabets Y_i and Y_{i+1} (corresponding, respectively, to some parts Y'_j and Y''_j of the tape alphabet of the machine M_3).

A chain \mathbf{A} is *non-trivial* if it has at least one cell.

Lemma 8.1. *Let \mathbf{x} be a subpath of the boundary of a reduced diagram Δ without hubs, and $\text{Lab}(\mathbf{x})$ a reduced word in a tape subalphabet Y_i of the machine M . Then no chain \mathbf{A} can start and end on \mathbf{x} .*

Proof. Proving by contradiction, we assume that a non-trivial chain \mathbf{A} starts and ends on \mathbf{x} . Notice that it cannot cross itself since every cell has at most two a -edges by (iii) (a). Thus it starts with an edge e of \mathbf{x} and ends at an edge f of \mathbf{x} , where $e \neq f$, but $\text{Lab}(e) \equiv \text{Lab}(f)^{-1}$ since these two letters belong to the same subalphabet Y_i . We may assume that \mathbf{A} is chosen so that $e\mathbf{x}'f$ is a subpath of \mathbf{x} , where the subpath \mathbf{x}' is of minimal possible length.

Now the chain \mathbf{A} and the path \mathbf{x}' bound a subdiagram Δ' all of whose boundary a -edges belong to \mathbf{x}' . By our observation, every q -edge on the boundary $\partial\Delta'$ is a $p^{\pm 1}$ -edge for the same control base letter $p = p_j$. Also every p -cell π of Δ' is active from both sides. Indeed if a cell π of Δ' is not active from both sides it does not belong to the chain \mathbf{A} by definition. Therefore π must have a neighbor (θ, s_i) -cell in Δ' by ((iv)) (consider the maximal θ -band of Δ' containing π). Then we may apply Lemma 5.6 to the maximal s_i -band of Δ' containing the (θ, s_i) -cell and obtain a letter of the form s_i in the boundary label of \mathbf{A} or in $\text{Lab}(\mathbf{x})$, a contradiction.

It follows that every (θ, q) -cell in Δ' is a p -cell (corresponds to a control letter) active from both sides. Since a maximal chain cannot end on a p -cell, every maximal chain of Δ' must start and end on \mathbf{x}' .

This property and the minimality of choice for \mathbf{x}' imply that $|\mathbf{x}'| = 0$, and so $\text{Lab}(\mathbf{x})$ has a non-empty freely trivial subword $\text{Lab}(ef)$; a contradiction. The lemma is proved. \square

Lemma 8.2. *Let a non-trivial chain \mathbf{A} crosses a maximal p -band \mathcal{C} of a reduced diagram Δ over M twice at cells π' and π'' . Then there is a p -cell π_0 in \mathcal{C} between π' and π'' , which is not active from both sides.*

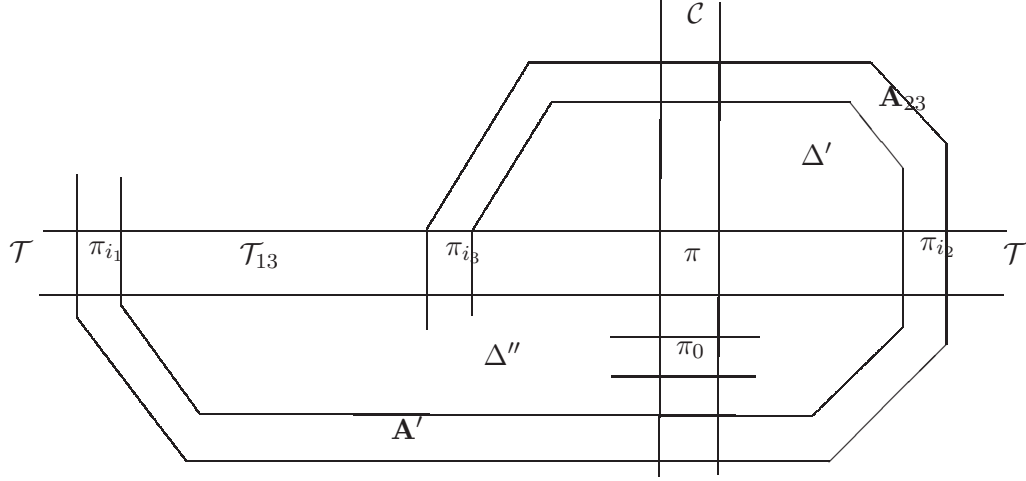
Proof. As in the proof of Lemma 8.1, there is a subdiagram Δ' bounded by the portion of \mathbf{A} and a segment \mathbf{x} of the side of \mathcal{C} between π and π' . As there, arguing by contradiction, we may assume that \mathbf{x} has no a -edges. Hence, if there is no cell π_0 lying between π' and π'' in \mathcal{C} which is not active from both sides then the cells π and π' must share a p -edge. Since the chain \mathbf{A} connects π and π' , we obtain that these two cells must have the same a -letter in the boundary labels (being letters from the same subalphabet Y_i), and they are mirror copies of each other as it follows from the defining relations of the group M , having two a -letters.

We come to a contradiction because Δ is a reduced diagram. The lemma is proved. \square

Lemma 8.3. *Assume that no θ -band of a comb Δ has a base with a subword $(pp^{-1}p)^{\pm 1}$, where p is a control letter. Then every chain \mathbf{A} of Δ has at most 9 common (θ, a) -cells with any θ -band \mathcal{T} of Δ .*

Proof. Let π_1, \dots, π_m be the common cells of \mathbf{A} and \mathcal{T} counted on \mathcal{T} from left to right. Denote by \mathcal{T}' the subband of \mathcal{T} starting with π_1 and ending by π_m . Every (θ, q) -cell of \mathcal{T}' must be a p -cell since every maximal q -band of Δ crossing \mathcal{T}' , has to cross the chain \mathbf{A} too by Lemma 5.6. Since by the assumption of the lemma and by Property (i), the base of \mathcal{T} has no triples of consecutive $p^{\pm 1}$ -letters, the subband \mathcal{T}' can have at most two (θ, q) -cells.

Assume that \mathbf{A} crosses \mathcal{T} consequently at 3 (θ, a) -cells $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}$, and $i_1 < i_3 < i_2$, i.e., \mathbf{A} has a ‘convolution of a spiral’ \mathbf{A}' , starting at π_{i_1} and ending at π_{i_3} .



By Lemma 8.1 for the part Δ' of Δ bounded by the subchain \mathbf{A}_{23} of \mathbf{A}' connecting π_{i_2} and π_{i_3} and the subband of \mathcal{T} connecting the same cells, \mathcal{T} has a p -cell π between π_{i_2} and π_{i_3} . The maximal p -band \mathcal{C} crossing \mathcal{T} at π must cross \mathbf{A}' at least twice (above and below \mathcal{T}). Hence there is a part of \mathbf{A}' satisfying, together with \mathcal{C} , the condition of Lemma 8.2, and so there is a cell π_0 given by that lemma, inside the part Δ'' of Δ bounded by \mathbf{A}' and the part \mathcal{T}_{13} of \mathcal{T} connected π_{i_1} and π_{i_3} . But then, according to Property ((iv)), Δ'' has to contain some s_i -cell neighboring π_0 , contrary to Lemma 5.6 since neither \mathbf{A}' nor \mathcal{T}_{13} have s_i -cells. Thus our assumption on the existence of \mathbf{A}' leads to a contradiction, and so $i_3 > i_2$ if $i_2 > i_1$.

Assume that \mathbf{A} crosses \mathcal{T} consequently at 4 (θ, a) -cells $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$. We may assume that $i_1 < i_2$, and so $i_1 < i_2 < i_3 < i_4$. Then again by Lemma 8.1, we have at least 3 p -cells on \mathcal{T}' (between π_{i_u} and $\pi_{i_{u+1}}$ for $u = 1, 2, 3$), a contradiction. Hence such a series of common (θ, a) -cells is impossible, and since \mathbf{A} has at most 2 common (θ, q) -cells with \mathcal{T} , we conclude that traveling along \mathbf{A} one meets $m \leq 3 + 1 + 3 + 1 + 3 = 11$ cells of \mathcal{T} (at most 3 (θ, a) -cells, then a (θ, q) -cell, and so on), and the number of a -cells among them does not exceed 9. \square

Lemma 8.4. *A comb Δ has no chain-annuli.*

Proof. Assume that \mathbf{A} is a chain-annulus. By Lemma 5.6 it must have a link π_i . Then there is a left-most maximal $p_i^{\pm 1}$ -band \mathcal{C}' of the comb Δ containing a link π_i from \mathbf{A} , where p_i is a control letter, i.e., the subcomb Γ of Δ with handle \mathcal{C}' has no links of \mathbf{A} except for those on \mathcal{C}' . But $\partial\pi_i$ has an a -edge from the left by (iii), and so two different a -edges of \mathcal{C}' are connected by an a -band \mathcal{A}_j from the left of \mathcal{C}' . We will assume that \mathcal{A}_j

is the shortest a -band with this property, and so the (θ, q) -cells π^1, \dots, π^m situated on \mathcal{C}' between π_{j-1} and π_j (if any) are inside the subdiagram Δ' bounded by the chain \mathbf{A} . By Lemma 5.6 for Δ' , these cells cannot have common edges with (θ, q) -cells which do not correspond to $p_i^{\pm 1}$. Now it follows from (iv) that the cells π^1, \dots, π^m are active from both sides. Since the cells π_{j-1} and π_j of the chain-annulus are also active from both sides, the existence of the a -band \mathcal{A}_j contradicts Property (vii). The lemma is proved. \square

8.2 An application of quasicombs

The surgery we use in Lemma 8.6 requires a slight modification of the notion of comb.

Let Δ be a reduced diagram over M with boundary \mathbf{yz} such that every maximal θ -band of Δ has exactly one θ -edge from \mathbf{y} . Assume that one can construct a q -band \mathcal{Q} with a top or bottom path \mathbf{y}_0 , and $\text{Lab}(\mathbf{y})$ can be obtained from $\text{Lab}(\mathbf{y}_0)$ after deleting of some a -letters. Then we say that Δ is a *quasicomb* with the *support* \mathbf{y} . As for combs, we use the standard factorization $\mathbf{yz} = \mathbf{y}^\Delta \mathbf{z}^\Delta$ of the boundary path of a quasicomb. The number $h = h^\Delta$ is the θ -length $|\mathbf{y}|_\theta$; the notation $[\Delta]$ is also extended to quasicombs as well as h_- , and the κ -, λ -, μ -, and ν -mixtures.

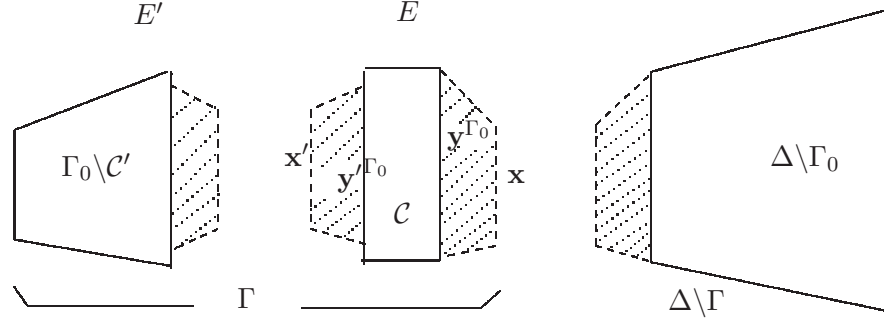
In particular, every comb is a quasicomb. (Take \mathcal{Q} to be the handle of the comb.) It follow from the definition that the history H of the quasicomb Δ is the history of \mathcal{Q} , and by Lemma 5.6, the boundary label of Δ uniquely determines the end edges of all maximal θ -bands $\mathcal{T}_1, \dots, \mathcal{T}_h$ starting on \mathbf{y} . It is easy to see that the set of cells of \mathcal{T}_1 is also uniquely determined by the boundary of Δ , and by induction, the same is true for $\mathcal{T}_2, \dots, \mathcal{T}_h$. Hence every quasicomb (in particular, every comb) is a minimal diagram.

Remark 8.5. The statements and the proofs of Lemmas 8.3 and 8.4 remain valid if Δ is a quasicomb.

We say that a diagram Δ *admits* a (proper) quasicomb Γ if it has a (proper) subcomb Γ_0 such that $\text{Lab}(\mathbf{z}^\Gamma) \equiv \text{Lab}(\mathbf{z}^{\Gamma_0})$, the handle of Γ_0 serves for Γ as \mathcal{Q} in the definition of quasicomb, and the words $\text{Lab}(\mathbf{y}^\Gamma)$ and $\text{Lab}(\mathbf{y}^{\Gamma_0})$ are equal modulo (θ, a) -relations. In particular, every subcomb of Δ is admitted.

Lemma 8.6. *Let Δ be a comb with history of type (2). Assume that Δ has a subcomb Γ_0 of base width ≤ 3 with a $s_j^{\pm 1}$ -handle \mathcal{C}' such that the filling trapezium $\text{Tp}(\mathcal{C}', \mathcal{C})$ is not aligned. Also assume that all maximal q -bands of Γ_0 except for \mathcal{C}' , are $p^{\pm 1}$ -bands. Then Δ admits a proper quasicomb Γ such that $\text{Area}(\Gamma) \leq 5(\delta')^{-1}[\Gamma]$.*

Proof. By Property (xiii), $\text{Lab}(\mathbf{y}^{\Gamma_0})$ has a factorization of the form $u(b_1 v_1 b_1^{-1}) \dots (b_m v_m b_m^{-1}) w$, where $b_i^{\pm 1}$ is an a -letter or 1 ($i = 1, \dots, m$), v_i is a group word in θ -letters, b_i commutes with every letter of v_i by virtue of (θ, a) -relations, and each of u, w has at most one a -letter. Similar property holds for $\text{Lab}(\mathbf{y}^{\Gamma_0})$. Hence one can separate the band \mathcal{C}' from the diagram Γ_0 and attach several (θ, a) -cells to the left and to the right sides of it, and obtain an auxiliary subdiagram E with the boundary \mathbf{exfx}' whose label 'almost' equal to the boundary label of \mathcal{C}' , but $\text{Lab}(\mathbf{x})$ (resp., $\text{Lab}(\mathbf{x}')$) are obtained from $\text{Lab}(\mathbf{y}^{\Gamma_0})$ (from $\text{Lab}(\mathbf{y}^{\Gamma_0})$) by deleting of the a -letters $b_i^{\pm 1}$ -s, and therefore each of \mathbf{x} and \mathbf{x}' has at most 2 a -edges.



Then we continue the surgery as follows. We can construct the mirror copies of the (θ, a) -cells attached to \mathbf{y}^{Γ_0} in E , and attach these copies to the diagram $\Gamma_0 \setminus C'$ to obtain (after possible cancellations of some (θ, a) -cells) a reduced quasicomb E' whose support can be denoted by \mathbf{x}' since its label is $\text{Lab}(\mathbf{x}')$, and the boundary of E' is $\mathbf{x}'(\mathbf{z}')^{\Gamma_0}$. Finally, we identify E' and E along \mathbf{x}' and (after possible cancellations of (θ, a) -cells) we have a desired quasicomb Γ with support \mathbf{x} . Now, to estimate the area of the minimal diagram Γ from above it suffices to estimate $\text{Area}(E')$ (and use Lemma 5.2) since obviously $\text{Area}(E) \leq 3(\text{Area}(C')) \leq 3h$.

Since the path \mathbf{x} has at most 2 a -edges we obtain

$$h^{\Gamma_0} \leq |\mathbf{x}| \leq h^{\Gamma_0} + 2\delta' \quad (8.15)$$

by the definition of the length of a path.

Note that at most two non-trivial chains of E' can start/end on \mathbf{x}' since $|\mathbf{x}'|_a \leq 2$. Consequently, by Lemma 8.3 (and Remark 8.5), every θ -band of E' has at most 18 cells belonging to such chains.

Let d be the maximal number of a -cells in θ -bands of E' . Taking into account the argument of the previous paragraph and the lack of chain-annuli in E' , we conclude that there are at least $(d - 18)/9$ maximal chains having both ends on $(\mathbf{z}')^{\Gamma_0}$. Now it follows from Lemma 5.20 and Inequality (8.15) that

$$|\mathbf{z}^\Gamma| \geq |(\mathbf{z}')^{\Gamma_0}| + 2 \geq 2 + h^{\Gamma_0} + 2\delta'(d - 18)/9 \geq 2 + |\mathbf{x}| - 2\delta' + 2\delta'(d - 18)/9 > |\mathbf{x}| + 1 + 2\delta'(d + 2)/9$$

because $\delta' \leq 1/7$. This implies $d + 2 \leq 5(\delta')^{-1}(|\mathbf{z}^\Gamma| - |\mathbf{x}| - 1)$, and since every θ -band of E' has at most 2 p -cells, we have, as required:

$$\text{Area}(\Gamma) \leq \text{Area}(E') + \text{Area}(E) \leq (2 + d)h^\Gamma + 3h^\Gamma < 5h^\Gamma(\delta')^{-1}(|\mathbf{z}^\Gamma| - |\mathbf{x}|)$$

□

Lemma 8.7. *Let Δ be a comb with base width $b \geq N$. Assume that Δ has a one-Step history and has no active k - or k' -cells. Then Δ admits a proper quasicomb Γ such that $\text{Area}(\Gamma) \leq 5(\delta')^{-1}[\Gamma]$.*

Proof. We first assume that the history of Δ is of type (2). It follows from the conditions of the lemma and (ii) that Δ cannot have k - or k' -cells at all, with the possible exception for the cells of the first and the last $(12)^{\pm 1}$ or $(23)^{\pm 1}$ -bands. By Lemma 7.20, we may also assume that the base of any θ -band of Δ does not have subword $(pp^{-1}p)^{\pm 1}$, where $p^{\pm 1}$ is any control letter.

Therefore a left-most maximal $s_i^{\pm 1}$ -, $k^{\pm 1}$ -, $(k')^{\pm 1}$ -, $t^{\pm 1}$ -, or $(t')^{\pm 1}$ -band \mathcal{C}' of Δ is a handle of a subcomb Γ of base width $b^\Gamma \leq 3$ and all other q -bands of Γ (if any exist) correspond to the same control letter $p^{\pm 1}$ by (i). One may assume that \mathcal{C}' has $h' > (\delta')^{-1}$ cells because otherwise Lemma 7.15 is applicable to the subcomb with handle \mathcal{C}' .

The filling trapezium $\Delta' = Tp(\mathcal{C}', \mathcal{C})$ has height $h' > (\delta')^{-1} > 2$, and therefore it has a maximal θ -band without k - or k' -cells. Hence the base of Δ' has no k - or k' -letters. Since $b \geq N$, Δ' is of base width $b' \geq N - 2$, but having no k - or k' -letters, the base of Δ' is not aligned or has only $t^{\pm 1}$ or $(t')^{\pm 1}$ -letters by Remark 5.16. In the former case, we are done by Lemma 8.6. In the later case we may refer to Lemma 7.18 since there are no derivative $p^{\pm 1}$ -bands for a t -band by Property (i).

If the history of Δ is of type (1) or (3), then it follows from the absence of active k - and k' -cells in Δ that all q -cells of Δ are passive by Property (ii), and the statement follows from Lemma 7.18. □

9 Combs with one-Step histories

In this section we obtain the estimates of the areas of one-Step combs. Lemmas 9.2, 9.6, 9.7 will be used in the next sections.

Lemma 9.1. *Let Γ be a comb whose proper subcombs have no active k - or k' -cells and the handle \mathcal{C} of Γ is passive from the left. Assume that Γ has one-Step history and admits no proper quasicombs Δ such that $\text{Area}(\Delta) \leq 5(\delta')^{-1}[\Delta]$. Assume that Γ has at most L_0 odd maximal θ -bands for some L_0 . Then (a) $\text{Area}'(\Gamma) \leq 5(\delta')^{-1}h(|\mathbf{z}'| - |\mathbf{y}'| + L_0)$; (b) if \mathcal{C} is also passive from the right, then $\text{Area}(\Gamma) < 5(\delta')^{-1}h(|\mathbf{z}| - |\mathbf{y}| + L_0)$;*

Proof. (a) By Lemma 8.7, we may assume that the base width b of Γ does not exceed N . By Lemma 7.20, we may assume that Δ has no bands with $p^{\pm 1}p^{\mp 1}p^{\pm 1}$ in the base for arbitrary control letter p . By Lemma 8.3, every chain of Γ and every θ -band of Γ have at most 9 a -cells in common. We also use below that Γ has no chain-annuli by Lemma 8.4.

At most $2(b - 1)L_0$ non-trivial maximal chains can start/terminate on the odd θ -bands of Γ by (iii) (a) - (c). Let \mathcal{T} be a θ -band of Γ having maximal number of a -cells d . Then among maximal chains crossing \mathcal{T} , we have at least $(d - 18(b - 1)L_0)/9$ chains with both ends on \mathbf{z}' , and so $|\mathbf{z}'|_a \geq 2(d - 18(b - 1)L_0)/9$. Therefore by Lemma 5.20,

$$|\mathbf{z}'| - |\mathbf{y}'| \geq 2(b - 1) + 2\delta'(d - 18(b - 1)L_0)/9 > 2\delta'(d + b - 1)/9 - 4\delta'NL_0$$

since $b \leq N$. Hence $d + b - 1 \leq (9/2)(\delta')^{-1}(|\mathbf{z}'| - |\mathbf{y}'|) + 18NL_0$, and therefore

$$\text{Area}'(\Gamma) \leq (b - 1 + d)h \leq 5h(\delta')^{-1}(|\mathbf{z}'| - |\mathbf{y}'| + L_0)$$

since $18N \leq (\delta')^{-1}/2$.

(b) Since \mathcal{C} is passive from the right, we have $|\mathbf{y}| \leq |\mathbf{y}'| + 2\delta'$ and $|\mathbf{z}| = |\mathbf{z}'| + 2$. Therefore $|\mathbf{z}| - |\mathbf{y}| > |\mathbf{z}'| - |\mathbf{y}'| + 1$, and so statement (a) implies (b) because $\text{Area}(\mathcal{C}) \leq h$. □

Lemma 9.2. *Let Γ be a one-Step comb of base width $b \leq 15N$. Assume that its handle \mathcal{C} is a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band. Then*

(1) Γ has a subcomb Δ such that its handle \mathcal{C}^Δ is passive from the right, and

$$\text{Area}(\Delta) \leq c_1([\Delta] + \frac{1}{2}h^\Delta h_-^\Delta) \quad (9.16)$$

(2) $\text{Area}(\Gamma) \leq c_1([\Gamma] + \kappa^c(\Gamma))$.

Proof. (1) We may assume that Γ has no $t^{\pm 1}$ or $(t')^{\pm 1}$ -bands except for \mathcal{C} , since otherwise we obtain a smaller subcomb Δ , and Δ also satisfies the assumptions of the lemma. In particular, the derivative bands $\mathcal{C}_1, \dots, \mathcal{C}_s$ are all k^{-1} - or k'^{-1} -bands by Property (i).

Assume first that the history of Γ is of Step (1) or (3), and the derivative bands are not active from the right. Then either Γ has a maximal k - or $(k')^{-1}$ -band \mathcal{C}' active from the left and passive from the right (and there exists the left-most band with this property), or all its q -bands are passive by Properties (i) and (ii). In any case Lemma 7.18 (b) is applicable to a subcomb of Γ . Thus we may further assume that the history of Γ is of Step 2. Property (v) implies that the derivative bands of Γ are active from the right.

The sum of areas of derivative subcombs $\Gamma_1, \dots, \Gamma_s$ is at most $\sum (60Nh_i^2 + 2\alpha h_i)$ by Lemma 7.13, where $\alpha = |\mathbf{z}|_a$. Hence by Property (vii) and Lemma 7.10,

$$\text{Area}(\Gamma) \leq h(h_- + 3\alpha + 1) + 60N \sum h_i^2 \quad (9.17)$$

We also recall that by Lemma 7.10, $\alpha \leq (\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}| - 1)$ since $|\mathbf{y}| = h$.

We assume first that $h_i \leq 2 \sum_{j \neq i} h_j$ for every i . Then $h_- \geq \frac{1}{3} \sum h_i$, and therefore $\sum h_i^2 \leq h \sum h_i \leq 3hh_-$. Now it follows from (9.17) and the subsequent estimate for α that

$$\text{Area}(\Gamma) \leq (60N)3hh_- + hh_- + 3(\delta')^{-1}[\Gamma] \leq c_1[\Gamma] + c_1hh_-/2,$$

by the choice of c_1 , as required.

Now assume that there is i_0 such that $h_{i_0} > 2 \sum_{j \neq i_0} h_j$. Then less than $h_{i_0}/2$ maximal a -bands starting on \mathcal{C}_{i_0} end on other derivative bands, and we can refer to Lemma 7.17 (b). Therefore $\sum \text{Area}(\Gamma_i) \leq (\delta')^{-2}[\Gamma]$. Hence

$$\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] + h(h_- + 3\alpha + 1) \leq (\delta')^{-2}[\Gamma] + hh_- + 3(\delta')^{-1}[\Gamma] \leq c_1([\Gamma] + hh_-/2),$$

as required, since $c_1 > (\delta')^{-2} + 3(\delta')^{-1}$. Thus the desired inequality is true in any case.

(2) We will induct on the number of maximal q -bands in Γ . By (1), we have a subcomb Δ of Γ satisfying (9.16), and therefore

$$\text{Area}(\Delta) \leq c_1[\Delta] + c_1\kappa^c(\Delta) \quad (9.18)$$

by Lemma 7.5.

One may assume that the subcomb Δ of Γ is proper since otherwise it is nothing to prove. Now Γ is a union of Δ and the remaining comb Δ' with the handle \mathcal{C} . We observe that

$$\mathbf{y}^{\Delta'} = \mathbf{y} \text{ and } |\mathbf{z}^{\Delta'}| = |\mathbf{z}| - |\mathbf{z}^\Delta| + |\mathbf{y}^\Delta| \quad (9.19)$$

by Lemma 5.20 (b). By the inductive hypothesis,

$$\text{Area}(\Delta') \leq c_1[\Delta'] + c_1\kappa^c(\Delta') \quad (9.20)$$

Now the sum of the first summands in the right-hand sides of (9.18) and (9.20) does not exceed $c_1 h(|\mathbf{z}| - |\mathbf{y}|)$ because $(|\mathbf{z}^{\Delta'}| - |\mathbf{y}^{\Delta'}|) + (|\mathbf{z}^{\Delta}| - |\mathbf{y}^{\Delta}|) = |\mathbf{z}| - |\mathbf{y}|$ by (9.19). The sum of second summands of (9.18) and (9.20) does not exceed $c_1 \kappa^c(\Gamma)$ by Lemma 7.3(c), and the lemma is proved since $\text{Area}(\Gamma) = \text{Area}(\Delta) + \text{Area}(\Delta')$. \square

Let Γ be a comb with a handle \mathcal{C} . Assume that Γ is a subcomb of (or can be embedded as a subcomb in) a larger comb $\bar{\Gamma}$ with a handle $\bar{\mathcal{C}}$, and the filling trapezium is $Tp(\mathcal{C}, \bar{\mathcal{C}})$. Then the comb $\bar{\Gamma}$ is called an *extension* of Γ . The extension is called *regular* if the base width of $Tp(\mathcal{C}, \bar{\mathcal{C}})$ is at least N . A comb Γ is called *regular* if there exists a regular extension of Γ . Recall that by definition of comb, every cell (of the handle) of a subcomb of Γ is connected with \mathcal{C} by a θ -band. Therefore a subcomb of a regular comb is regular itself.

Remark 9.3. A regular comb is organized better than a random one because its history coincides with the history of the filling *trapezium* $Tp(\mathcal{C}, \bar{\mathcal{C}})$ having a sufficiently long base. Therefore this history is a subject of some restrictions imposed on trapezia by Proposition 5.17. Recall that by Lemma 5.10, the properties of trapezia reflect all the features of the computations executed by the machine M . In particular, the next lemma uses Property (xiv) based on the aperiodicity of the histories formulated in Lemma 4.29, which, in turn goes back to Lemmas 4.15 and 2.2(a).

Lemma 9.4. *Let Γ be a one-Step regular comb, and the handle \mathcal{C} of Γ active from the left k - or $(k')^{-1}$ -band. Assume that Γ has neither $t^{\pm 1}$ - nor $(t')^{\pm 1}$ -bands and has no active from the left maximal k - or $(k')^{-1}$ -bands except for \mathcal{C} , and Γ admits no proper quasicomb Δ such that $\text{Area}(\Delta) \leq 5(\delta')^{-1}[\Delta]$. Then $\text{Area}(\Gamma) \leq 16(\delta')^{-2}[\Gamma]$.*

Proof. Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_s$ be the set of derivative subbands (connected with \mathcal{C} by simple θ -bands). Since none of them is a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band, they must be k^{-1} - or k' -bands active from the right by (i). Moreover, since the derivative subcombs $\Gamma_1, \dots, \Gamma_s$ has no k - or k' -cells active from the left, it is easy to see from Property (i) that Γ has no active $k^{\pm 1}$ - or $(k')^{\pm 1}$ - cells except for those belonging to $\mathcal{C}, \mathcal{C}_1, \dots, \mathcal{C}_s$. By (vii), every maximal a -band starting on \mathcal{C} ends either on some \mathcal{C}_i or on $\mathbf{z} = \mathbf{z}^\Gamma$. Besides, if \mathcal{T} is a simple θ -band starting on \mathcal{C} , then every maximal a -band starting on \mathcal{T} ends either on one of the bands $\mathcal{C}, \mathcal{C}_1, \dots, \mathcal{C}_s$, or on \mathbf{z} .

Assume that \mathcal{T} is a simple θ -band of maximal length d among the simple θ -bands starting on \mathcal{C} . Then the total number of cells in all simple θ -bands is at most dh .

Let T_1 and T_2 be top and bottom of \mathcal{T} . Since no non-trivial a -band starts and ends on \mathcal{C} , either no a -band \mathcal{A} starting on T_1 (and having no cell from \mathcal{T}) ends on \mathcal{C} or no a -band \mathcal{A} starting on T_2 (and having no cells from \mathcal{T}) ends on \mathcal{C} . We consider only the first variant.

If there are m maximal θ -bands above \mathcal{T} , then at least $d - 1$ maximal a -bands starting on \mathcal{T} and $m - 1$ maximal a -bands starting on \mathcal{C} end on at most m (q, θ) -cells of the derivative bands and on \mathbf{z} . Therefore at least $d - 2$ of them end on \mathbf{z} , and so $|\mathbf{z}| - |\mathbf{y}| \geq 2 + (d - 2)\delta'$ by Lemma 5.20 (a). We set $\Delta_i = Tp(\mathcal{C}_i, \mathcal{C})$ ($i = 1, \dots, s$; and $d \geq 2$ if $s > 0$ since otherwise two neighbor k - or k' -cells of \mathcal{T} form a non-reduced subdiagram). By Property ((xv)), $H_i \equiv u_i w_i^{k_i} v_i$, where $|u_i|, |v_i| \leq (d - 1)/2$, $|w_i| \leq d - 1$.

Consider a regular extension $\bar{\Gamma}$ of the comb Γ . Let $\bar{\mathcal{C}}$ be a handle of $\bar{\Gamma}$ and $\Pi_i = Tp(\mathcal{C}_i, \bar{\mathcal{C}})$.

If the base of Π_i is not aligned, and Γ_i has an $s_j^{\pm 1}$ -band for some j , then it follows from Property (i) and Lemma 7.20 that Γ_i satisfies the assumptions of Lemma 8.6, and we have a contradiction with the hypothesis of Lemma 9.4. Therefore Γ_i has no $s_j^{\pm 1}$ -bands, and so, by Property (i), it has no maximal q -bands at all except for \mathcal{C}_i . Hence by Lemma 7.20, $\text{Area}'(\Gamma_i) \leq (\delta')^{-1}h_i(|\mathbf{z}'_i| - |\mathbf{y}'_i| + 1)$ in standard notation, if the base of Π_i is not aligned.

Assume that the base of Π_i is aligned. It starts with k (or with $(k')^{-1}$) and the second letter of base is not k^{-1} (not k') by Lemma 3.4. Therefore, by Property (i), the second letter is the copy of the first letter (or of the inverse of the last letter) of the standard base B of M_3 . Since the base of Π is aligned and its base width $\geq N > 2||B||$, this base is large. Therefore if $k_i \geq 3$, then the w^{k_i} -part of Π has no odd θ -bands by ((xiv)), and so the number of maximal odd θ -bands is at most $|u_i| + |v_i| \leq d - 1$. If $k_i \leq 2$, then $h_i \leq |u_i| + |w_i| + |v_i| \leq 3(d - 1)$. Thus, in any case, the number L_i of odd maximal θ -bands in Π_i (and in Γ_i) is at most $3(d - 1)$.

Using $\mathbf{z}_i, \mathbf{z}'_i, \mathbf{y}_i, \mathbf{y}'_i$ in the standard way for the subcombs Γ_i -s, we have $\text{Area}'(\Gamma_i) \leq 5(\delta')^{-1}h_i(|\mathbf{z}'_i| - |\mathbf{y}'_i| + L_i)$ by Lemma 9.1, if the base of Π_i is aligned. Thus in any case $\text{Area}'(\Gamma_i) \leq 5(\delta')^{-1}h_i(|\mathbf{z}'_i| - |\mathbf{y}'_i| + 3(d - 1))$. Since $|\mathbf{y}'_i| = h_i$ and $|\mathbf{y}| = h$, the sum of these areas is at most $5(\delta')^{-1}h(|\mathbf{z}| - |\mathbf{y}| + 3(d - 1))$. Finally,

$$\text{Area}(\Gamma) \leq dh + 5(\delta')^{-1}h(|\mathbf{z}| - |\mathbf{y}| + 3(d - 1)) \leq h(d + 5(\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}| + 3(d - 1))) \quad (9.21)$$

Recall that $d - 2 \leq (\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}|)$ and $|\mathbf{z}| - |\mathbf{y}| \geq 2$ since \mathbf{y} has no a -edges. Therefore

$$d + 5(\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}| + 3(d - 1)) \leq 16(\delta')^{-2}(|\mathbf{z}| - |\mathbf{y}|),$$

and so by (9.21), $\text{Area}(\Gamma) \leq 16(\delta')^{-2}[\Gamma]$, as required. \square

We call a (quasi)comb Γ *long* if $|\mathbf{z}| = |\mathbf{z}^\Gamma| > |\mathbf{y}| = |\mathbf{y}^\Gamma|$. If the handle of a comb Γ is passive from the right, then Γ is long since $|\mathbf{z}| \geq 2 + h$ and $|\mathbf{y}| = h$. Obviously a (quasi)comb Γ is long if $\text{Area}(\Gamma) \leq c[\Gamma]$ for some $c > 0$.

Remark 9.5. Observe that if Γ is a long subcomb of a diagram Δ then for the complementary subdiagram $\Delta' = \Delta \setminus \Gamma$ cut from Δ along the path \mathbf{y} , we have $|\partial\Delta'| < |\partial\Delta|$ by Lemma 5.20.

If Γ is a long quasicomb admitted by a minimal diagram Δ with boundary path $\mathbf{z}\mathbf{z}'$ (where $\mathbf{z} = \mathbf{z}^\Gamma$), then there exists a minimal diagram Δ' with boundary label $\text{Lab}(\mathbf{y}^{-1})\text{Lab}(\mathbf{z}')$. It follows from Lemma 5.6 that if Δ is a comb with handle \mathcal{C} , and Δ admits a proper quasicomb Γ , then Δ' is also a comb with the same handle \mathcal{C} but with fewer number of maximal q -bands than in Δ .

It is clear that $\text{Area}(\Delta) \leq \text{Area}(\Gamma) + \text{Area}(\Delta')$ (or see Lemma 5.2). We also use notation $\Delta \setminus \Gamma$ for such a ‘compliment’ Δ' of Γ . Then all the statements of Lemma 7.3 hold if one replaces ‘subcomb Γ ’ by ‘admitted quasicomb Γ ’ in their formulations because the quasicomb Γ and the subcomb Γ_0 from the definition of admitted quasicomb have equal κ -, λ -, μ -, and ν -necklaces.

Lemma 9.6. *Let Γ be a one-Step regular comb of base width $b \in [2N, 15N]$. Then Γ admits a long quasisubcomb Δ such that*

$$\text{Area}(\Delta) \leq c_1([\Delta] + \frac{1}{2}h^\Delta h_-^\Delta) \leq c_1([\Delta] + \kappa^c(\Delta))$$

Proof. For the beginning, we recall that every subcomb of a regular comb is regular. By Lemma 9.2 (and Lemma 7.5), one may assume that Γ has neither t - nor t' -cells. If Γ has a subcomb Δ of base width $\geq N$ without active $k^{\pm 1}$ - or $(k')^{\pm 1}$ -cells, then one can apply Lemma 8.7 since $c_1 \geq 5(\delta')^{-1}$. Otherwise by (ii), there is a maximal, active from the left or from the right q -band \mathcal{C}' corresponding to a $k^{\pm 1}$ - or $(k')^{\pm 1}$ -letter, such that the subcomb Δ with handle \mathcal{C}' has no other maximal active $k^{\pm 1}$ - or $(k')^{\pm 1}$ -bands and the base width of the filling trapezium $TP(\mathcal{C}', \mathcal{C})$, is at least N since $b \geq 2N$. If \mathcal{C}' is active from the left, then the statement follows from Lemma 9.4. Otherwise \mathcal{C}' is active from the right, and from the right of \mathcal{C}' , there must be a maximal band $\mathcal{C}'' \neq \mathcal{C}$ corresponding to k - or $(k')^{-1}$ -letter which therefore is active from the left. (Recall that Γ has neither t - nor t' -cells, and so such \mathcal{C}'' exists by Property (i).) Lemma 9.4 is now applicable to the subcomb with handle \mathcal{C}'' , and so Lemma 9.6 is proved in any case. \square

Lemma 9.7. *Let Γ be a regular comb having history of type (1) or (3) and base width at most $15N$. Assume that the handle \mathcal{C} is a passive from the left $k^{\pm 1}$ - or $(k')^{\pm 1}$ -, or s_0 -band. Then $\text{Area}'(\Gamma) \leq c_1 h(|\mathbf{z}'| - |\mathbf{y}'| + 1) + c_1 \kappa^c(\Gamma)$.*

Proof. At first we will prove that either Γ admits a proper quasicomb Δ such that

$$\text{Area}(\Delta) \leq c_1 [\Delta] + c_1 \kappa^c(\Delta) \quad (9.22)$$

or $\text{Area}'(\Gamma) \leq c_1 h(|\mathbf{z}'| - |\mathbf{y}'| + 1) + c_1 \kappa^c(\Gamma)$.

By Lemma 9.2, we may assume that Γ has neither $t^{\pm 1}$ -bands nor $(t')^{\pm 1}$ -bands, and it also has neither k -bands nor $(k')^{-1}$ -bands active from the left by Lemma 9.4. Therefore by Property (i), there are no active $k^{\pm 1}$ - or $(k')^{\pm 1}$ -cells except for those in \mathcal{C} . Since the step history of Γ is (1) or (3), every other q -bands of Γ is passive by (ii), and if there exist other q -bands, then we can find the desired subcomb Δ by Lemma 7.18(b). If Γ has no maximal q -bands except for the handle \mathcal{C} , then statement follows from Lemma 7.18(a).

To complete the proof, we will induct on the number of maximal q -bands in Γ , as in Lemma 9.2 (b). By the previous argument, we may assume that Γ admits a proper (quasi)subcomb Δ satisfying (9.22), since otherwise there is nothing to prove. Now Γ is a union of Δ and the ‘compliment’ $\bar{\Delta} = \Gamma \setminus \Delta$ (see Remark 9.5) with the handle \mathcal{C} . By the inductive hypothesis,

$$\text{Area}'(\bar{\Delta}) \leq c_1 h^{\bar{\Delta}}(|(\mathbf{z}')^{\bar{\Delta}}| - |(\mathbf{y}')^{\bar{\Delta}}| + 1) + c_1 \kappa_{1,1}^c(\bar{\Delta}) \quad (9.23)$$

Note that $h^{\bar{\Delta}} \leq h$. It follows that the sum of the first summands of (9.22) and (9.23) does not exceed $c_1 h(|\mathbf{z}'| - |\mathbf{y}'| + 1)$. The sum of second summands of (9.22) and (9.23) does not exceed $c_1 \kappa_{1,1}^c(\Gamma)$ by Lemma 7.3(c), and lemma is proved since $\text{Area}'(\Gamma) = \text{Area}(\Delta) + \text{Area}'(\bar{\Delta})$.

10 Combs with incomplete sets of Steps

In this section, we analyze combs whose histories have no rules of one of the Steps (1) or (3), and the main goal is Lemma 10.6. Lemmas 10.5 and 10.6 utilize the λ -mixture, but unfortunately, this parameter can be negative for some other combs. Therefore first of all we have to bound it from below in terms of other ‘quadratic’ parameters of combs.

Lemma 10.1. *Let Γ be a comb whose handle \mathcal{C} is either (a) a $t^{\pm 1}$ -band or (b) a $(t')^{\pm 1}$ -band. Respectively, let the history H of Γ , either (a) have rules $(23)^{\pm 1}$ but no rules $(12)^{\pm 1}$ or (b) have rules $(12)^{\pm 1}$ but no rules $(23)^{\pm 1}$. Then $\lambda^c(\Gamma) \geq -8(\delta')^{-1}[\Gamma] - 36\kappa^c(\Gamma)$.*

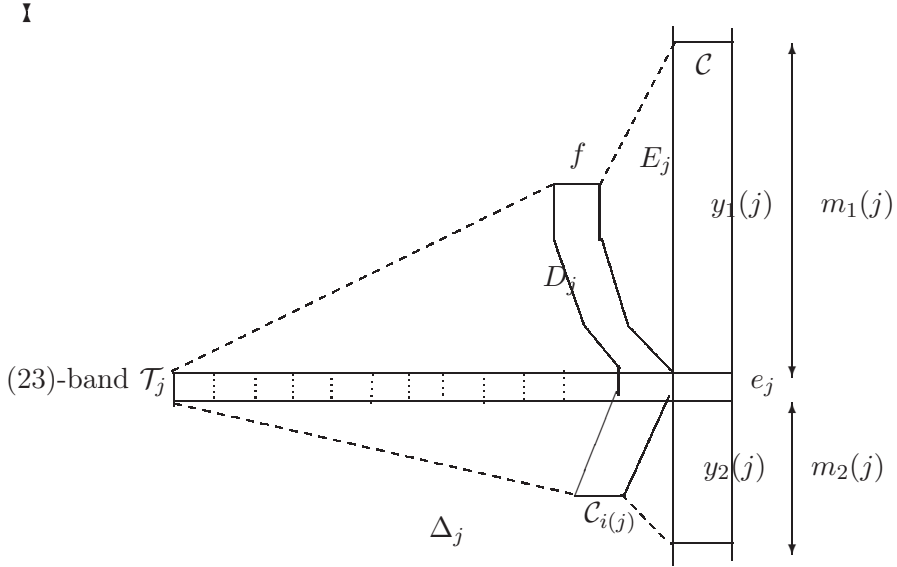
Proof. We shall prove the variant (a) only. For this goal we consider the set \mathbf{T} of maximal (23)-bands of Γ , which do not cross derivative bands of Γ , and so both their edges on \mathbf{y}^Γ and on \mathbf{z}^Γ are labeled by the same θ -letter, and therefore they are non-special edges by the definition of λ^c . (The set \mathbf{T} may be empty.) Consider all (non-empty) maximal combs Δ_j ($j = 1, \dots, r$) in which Γ is separated by the bands of \mathbf{T} . (The combs Δ_j -s do not contain the separating θ -bands from \mathbf{T} .) Then

$$\lambda^c(\Gamma) = \sum_j \lambda^c(\Delta_j) \quad (10.24)$$

because arbitrary two white beads which are not separated by a black one in $\mathbf{z} = \mathbf{z}^\Gamma$ or in $\mathbf{y} = \mathbf{y}^\Gamma$ belong to the boundary of some Δ_j since black beads are placed on q -edges and also on non-special θ -edges (and also, every bead of $\partial\Delta_j$ is on \mathbf{z}). Below we call an edge *black* (*white*) if its middle point is a black (white) bead.

Therefore we will estimate $\lambda^c(\Delta_j)$ for every j from below and then will use (10.24). Clearly, this number is at least $-\lambda^c(\mathbf{y}(j))$ ($j = 1, \dots, r$), where $\mathbf{y}(j) = \mathbf{y}^{\Delta_j}$. To give an upper bound for $\lambda^c(\mathbf{y}(j))$, we apply Lemma 6.2 (e) to \mathbf{y}_j and select an appropriate 'black' edge e_j (if any exists) on $\mathbf{y}(j)$ such that $\mathbf{y}(j) = \mathbf{y}_2(j)e_j\mathbf{y}_1(j)$ with $m_1(j)$ and $m_2(j)$ white beads on $\mathbf{y}_1(j)$ and $\mathbf{y}_2(j)$, respectively, and

$$\lambda^c(\mathbf{y}(j)) \leq 2m_1(j)m_2(j) \quad (10.25)$$



Since the path y has no q -edges, e_j is a (23)-edge. Let \mathcal{T}_j be the maximal (23)-band of Δ_j containing the edge e_j . By the choice of Δ_j , the θ -band \mathcal{T}_j crosses a derivative band $\mathcal{C}_{i(j)}$ of Γ , and this derivative band is a k^{-1} -band by Properties (i) and (v). The Step history of $\mathcal{C}_{i(j)}$ has no subwords $(23)^{-1}(2)(23)$ by (viii) applied to the filling trapezium $TP(\mathcal{C}_{i(j)}, \mathcal{C})$, and so the top or the bottom path of \mathcal{T}_j cuts the derivative band in two parts such that one of them is a k^{-1} -subband \mathcal{D}_j with Step history (2) (without the cell

from \mathcal{T}_j). We denote by $d(j)$ the length of this subband. Every maximal θ -band crossing a derivative band, also crosses the handle of Δ_j , so we may assume that \mathcal{D}_j belongs to the union E_j of the maximal θ -bands of Γ ending on the $m_1(j)$ white edges of \mathbf{z}^{Δ_j} , because one can interchange $m_1(j)$ and $m_2(j)$ in (10.25), in particular, $d(j) \leq m_1(j)$. We consider 3 cases.

(1) $d(j) \leq m_1(j)/2$. Then we say that $j \in J_1 \subset \{1, \dots, r\}$.

(2) $d(j) > m_1(j)/2$ and at least $d(j)/2$ maximal a -bands starting on \mathcal{D}_j end on derivative bands of Γ non-equal to $\mathcal{C}_{i(j)}$. Then $j \in J_2$.

(3) $d(j) > m_1(j)/2$ and less than $d(j)/2$ maximal a -bands starting on \mathcal{D}_j end on derivative bands non-equal to $\mathcal{C}_{i(j)}$. Then $j \in J_3$.

Case (1). In this case, the end f of \mathcal{D}_j is a q -edge factorizing the path $\mathbf{z} = \mathbf{z}^{\Delta_j}$ in a product $\mathbf{z}'f\mathbf{z}''$, where \mathbf{z}' has at least $m_1(j)/2$ white θ -edges (the ends of maximal θ -bands from E_j non-crossing \mathcal{D}_j) and \mathbf{z}'' has at least $m_2(j)$ white edges. Therefore $m_2(j)m_1(j)/2 \leq \kappa(\mathbf{z}^{\Delta_j}) = \kappa^c(\Delta_j)$. Hence by Inequality (10.25) and by Lemma 7.9 (a),

$$\sum_{j \in J_1} \lambda^c(\mathbf{y}(j)) \leq 2 \sum_{j \in J_1} m_1(j)m_2(j) \leq 4 \sum_{j \in J_1} \kappa^c(\Delta_j) \leq 4\kappa^c(\Gamma) \quad (10.26)$$

Case (2). In this case, at least $d(j)/2$ a -bands connect $\mathcal{C}_{i(j)}$ with a different derivative band. Therefore the number of a -bands connecting pairwise different derivative bands of Γ is at least $\frac{1}{4} \sum_{j \in J_2} d(j)$. On the other hand, the same number does not exceed $h_- = h_-^\Gamma$ by Lemma 7.7, whence $\sum_{j \in J_2} d(j) \leq 4h_-$. Since $m_1(j) \leq 2d(j)$, this inequality implies $\sum_{j \in J_2} m_1(j) \leq 8h_-$. Therefore by Lemma 7.7 and (10.25), we have

$$\sum_{j \in J_2} \lambda^c(\mathbf{y}(j)) \leq 2(8h_-) \sum_{j \in J_2} m_2(j) \leq 16h_-h \leq 32\kappa^c(\Gamma) \quad (10.27)$$

Case (3). Since the Step history of the k^{-1} -band \mathcal{D}_j is (2), every cell of this band is active from the right by (ii), and so there are no a -bands starting and ending on the same \mathcal{D}_j by (vii). No a -band starting on \mathcal{D}_j can cross \mathcal{T}_j since there are no (θ, a) -cells between the intersection cells of \mathcal{T}_j with $\mathcal{C}_{j(i)}$ and with \mathcal{C} by (v). Thus in Case (3), more than $d(j)/2$ a -bands starting on \mathcal{D}_j end on $\partial\Gamma$. Therefore $|\mathbf{z}|_a \geq \sum_{j \in J_3} d(j)/2 \geq \sum_{j \in J_3} m_1(j)/4$. Hence, by Lemma 7.10 (a),

$$\sum_{j \in J_3} \lambda^c(\mathbf{y}(j)) \leq 2 \sum_{j \in J_3} m_1(j)m_2(j) \leq 8|\mathbf{z}|_a h \leq 8(\delta')^{-1}[\Gamma] \quad (10.28)$$

Altogether, Inequalities (10.24) and (10.26)–(10.28) imply the inequality

$$\lambda^c(\Gamma) \geq - \sum_{j \in J_1 \cup J_2 \cup J_3} \lambda^c(\mathbf{y}(j)) \geq -36\kappa^c(\Gamma) - 8(\delta')^{-1}[\Gamma].$$

□

Lemma 10.2. *Let Δ be a regular comb with step history (2)(1)(2) or (2)(3)(2), where the first or the last (2) can be absent. Let $H \equiv H(1)H(2)H(3)$ be the corresponding Step decomposition of the history H . Assume that the handle \mathcal{C} is k^{-1} - or k' -, or s_0 -band and the base width of Δ is at most $15N$. Let $l = \|H(1)\| + \|H(3)\|$, then $\text{Area}(\Delta) \leq c_2(h(|\mathbf{z}'| - h + l + 1) + \kappa^c(\Delta))$, where $h = \|H\|$ and $\mathbf{z}' = \mathbf{z}'^\Delta$.*

Proof. Let $\Delta(i)$ be the $H(i)$ -part of Δ , $i = 1, 2, 3$. We will abbreviate $h(1) = h^{\Delta(1)}$, and so on. By (ii), the handle \mathcal{C} is passive from the left, and so by Lemma 9.7,

$$\text{Area}'(\Delta(2)) \leq c_1 h(2)(|\mathbf{z}'(2)| - h(2) + 1) + c_1 \kappa^c(\Delta(2)) \quad (10.29)$$

and therefore the statement is true if $l = 0$ since $c_2 > c_1 + 1$. Further we assume that $l \geq 1$.

Since the base widths of Δ does not exceed $15N$, we have by Lemma 7.13,

$$\text{Area}(\Delta(1)) + \text{Area}(\Delta(3)) \leq 60N(h(1)^2 + h(3)^2) + 2(h(1)\alpha(1) + h(3)\alpha(3)) \quad (10.30)$$

where $\alpha(1) = |\mathbf{z}(1)|_a$, $\alpha(3) = |\mathbf{z}(3)|_a$.

Let \mathbf{x}_{12} be the common part of the boundaries of $\Delta(1)$ and $\Delta(2)$. By Lemma 7.13,

$$|\mathbf{x}_{12}|_a \leq (\alpha(1) + 120Nh(1))/2 \leq (\alpha + |\mathbf{x}_{12}|_a + 120Nh(1))/2, \quad (10.31)$$

where $\alpha = |\mathbf{z}|_a = |\mathbf{z}^\Delta|_a$. From (10.31) and similar inequality with \mathbf{x}_{23} , we have

$$|\mathbf{x}_{12}|_a \leq \alpha + 120Nh(1) \text{ and } |\mathbf{x}_{23}|_a \leq \alpha + 120Nh(3) \quad (10.32)$$

Therefore

$$\alpha(1) \leq |\mathbf{x}_{12}|_a + \alpha \leq 2\alpha + 120Nh(1) \text{ and } \alpha(3) \leq 2\alpha + 120Nh(3) \quad (10.33)$$

Now we use Inequalities (10.30), (10.33), equality $h(1) + h(3) = l$, and Lemma 7.10 (a) to obtain inequality

$$\begin{aligned} \text{Area}(\Delta(1)) + \text{Area}(\Delta(3)) &\leq 60Nl^2 + 2l(2\alpha + 120Nl) \\ &\leq 60Nl^2 + 2l(2(\delta')^{-1}(|\mathbf{z}'| - h) + 120Nl) \end{aligned} \quad (10.34)$$

We also have $|\mathbf{z}'(2)| \leq |\mathbf{z}'| + |\mathbf{x}_{12}| + |\mathbf{x}_{23}| \leq |\mathbf{z}'| + 2\alpha + 120Nl$ by (10.32). Therefore

$$|\mathbf{z}'(2)| - h(2) \leq |\mathbf{z}'| + 2\alpha + 120Nl - (h - l) \leq |\mathbf{z}'| - h + 2\alpha + 121Nl.$$

Hence $|\mathbf{z}'(2)| - h(2) \leq (|\mathbf{z}'| - h)(2(\delta')^{-1} + 1) + 121Nl$ by Lemma 7.10 (a). Using this estimate we deduce from (10.29) and Lemma 7.9(b) that

$$\begin{aligned} \text{Area}'(\Delta(2)) &\leq c_1 h(2)(|\mathbf{z}'(2)| - h(2) + 1) + c_1 \kappa^c(\Delta(2)) \\ &\leq c_1 h((|\mathbf{z}'| - h)(2(\delta')^{-1} + 1) + 122Nl) + c_1 \kappa^c(\Delta) \end{aligned}$$

In turn, this inequality and (10.34) show that

$$\begin{aligned} \text{Area}(\Delta) &\leq \text{Area}(\Delta(1)) + \text{Area}(\Delta(3)) + \text{Area}'(\Delta(2)) + h \\ &\leq 60Nhl + 2h(2(\delta')^{-1}(|\mathbf{z}'| - h) + 120Nl) \\ &\quad + c_1 h((|\mathbf{z}'| - h)(2(\delta')^{-1} + 1) + 122Nl) + c_1 \kappa^c(\Delta) + h \\ &\leq c_2 h(|\mathbf{z}'| - h + l) + c_2 \kappa^c(\Delta) \end{aligned}$$

because $l \geq 1$, $|\mathbf{z}'| - h \geq 0$ and $c_2 \geq c_1 \max(3(\delta')^{-1}, 123N)$. The lemma is proved. \square

Lemma 10.3. *Let Γ be a regular comb of base width $b \leq 15N$. Assume that it has no one-Step long subcombs Δ with $\text{Area}(\Delta) \leq c_1[\Delta] + c_1 \kappa^c(\Delta)$. Let the history H of Γ either (a) have rules $(23)^{\pm 1}$ but no rules $(12)^{\pm 1}$ or (b) have rules $(12)^{\pm 1}$ but no rules $(23)^{\pm 1}$. Also assume that in case (a), \mathcal{C} is a $t^{\pm 1}$ -band, and in case (b), \mathcal{C} is a $(t')^{\pm 1}$ -band. Then $\text{Area}(\Gamma) \leq c_2(2(\delta')^{-1}[\Gamma] + 6\kappa^c(\Gamma)) \leq c_3[\Gamma] + c_2\mu^c(\Gamma)$.*

Proof. Case (b): We consider the system of derivative bands $\mathcal{C}_1, \dots, \mathcal{C}_s$. As usual, Γ_i is the derivative subcomb with handle \mathcal{C}_i . If a derivative band is a $(t')^{\mp 1}$ -band, then it must have a one-step history since the base of a (12)-band cannot contain subwords $(t')^{\mp 1}(t')^{\pm 1}$ by (v). By Lemma 9.2 (2), this would contradict our assumption on long subcombs. Therefore all the derivative bands are k' -bands by (i).

Let them have histories H_1, \dots, H_s , respectively. Then H_i ($i = 1, \dots, s$) has no subwords of type $(12)(2)(12)^{-1}$ by (viii) applied to the filling trapezium $Tp(\mathcal{C}_i, \mathcal{C})$ (a similar argument was used in Lemma 10.1). Therefore $H_i \equiv H_i(1)H_i(2)H_i(3)$ ($i \leq s$), where $H_i(1)$ and $H_i(3)$ are of Step (2) and $H_i(2)$ is of Step (1), where some of the factors can be empty. The lengths of the histories are denoted by h_i and $h_i(j)$, $j = 1, 2, 3$.

If $H_i(2)$ is non-empty, then by Lemma 10.2 for Γ_i and equality $|\mathbf{z}|_i = |\mathbf{z}'|_i + 2$,

$$\text{Area}(\Gamma_i) \leq \text{Area}'(\Gamma) + h_i \leq c_2(h_i(|\mathbf{z}| - h_i + h_i(1) + h_i(3)) + \kappa^c(\Gamma_i))$$

If $H_i = H_i(1)$, then $\text{Area}(\Gamma_i) \leq 60N h_i(1)^2 + 2h_i(1)|\mathbf{z}|_a$ by Lemma 7.13.

Since $c_2 \geq 2(\delta')^{-1}$ and $|\mathbf{z}|_a \leq (\delta')^{-1}(|\mathbf{z}| - h)$ by Lemma 5.20 (a), we derive from the inequalities of the previous paragraph and Lemma 7.9(b) that

$$\sum_{i=1}^s \text{Area}(\Gamma_i) \leq c_2 h(|\mathbf{z}| - h + \sum (h_i(1) + h_i(3))) + c_2 \kappa^c(\Gamma) + 60N \sum h_i(1)^2 \quad (10.35)$$

Let $h_{i_0} = \max h_i$. Then $\sum_{i \neq i_0} h_i = h_-$ by the definition of h_- . By (vii), every maximal a -band starting on the $H_{i_0}(1)$ - or $H_{i_0}(3)$ - part of \mathcal{C}_{i_0} must end either on the $H_i(1)$ - or $H_i(3)$ -part of some \mathcal{C}_i , $i \neq i_0$, or on the path \mathbf{z} . (Indeed, the $H_i(1)$ -part of a band \mathcal{C}_i cannot be connected with the $H_i(3)$ -part by an a -band since the $H_i(2)$ -part of \mathcal{C}_i has common θ -edges with \mathcal{C} by (v).) Therefore $h_{i_0}(1) + h_{i_0}(3) \leq h_- + |\mathbf{z}|_a \leq h_- + (\delta')^{-1}(|\mathbf{z}| - h)$ by Lemma 5.20 (a), and so, $\sum_{i=1}^s (h_i(1) + h_i(3)) \leq 2h_- + (\delta')^{-1}(|\mathbf{z}| - h)$. It follows from this estimate, Inequalities (10.35) and $h h_- \leq 2\kappa^c(\Gamma)$ (see Lemma 7.5) that

$$\begin{aligned} \sum_{i=1}^s \text{Area}(\Gamma_i) &\leq c_2 h((|\mathbf{z}| - h)(1 + (\delta')^{-1}) + 2h_-) + c_2 \kappa^c(\Gamma) + 60N h(2h_- + (\delta')^{-1}(|\mathbf{z}| - h)) \\ &\leq (c_2 + c_2(\delta')^{-1} + 60N(\delta')^{-1})[\Gamma] + (4c_2 + c_2 + 240N)\kappa^c(\Gamma) \end{aligned} \quad (10.36)$$

By Lemmas 7.10 and 7.5, the total area of all simple bands in Γ does not exceed

$$h(h_- + |\mathbf{z}|_a + 1) \leq h(h_- + (\delta')^{-1}(|\mathbf{z}| - h - 1)) < (\delta')^{-1}[\Gamma] + 2\kappa^c(\Gamma)$$

This inequality and (10.36) imply

$$\text{Area}(\Gamma) \leq \left(\frac{3}{2}(\delta')^{-1}c_2 + 60N(\delta')^{-1}\right)[\Gamma] + (5c_2 + 240N + 2)\kappa^c(\Gamma) \quad (10.37)$$

and to obtain inequality $\text{Area}(\Gamma) \leq c_2(2(\delta')^{-1}[\Gamma] + 6\kappa^c(\Gamma))$, it remains to use that $c_2 > 240N + 2$.

To complete the proof, we observe that by Lemma 10.1,

$$\begin{aligned} c_3[\Gamma] + c_2\mu^c(\Gamma) &= c_3[\Gamma] + c_2(c_0\kappa^c(\Gamma) + \lambda^c(\Gamma)) \\ &\geq c_3[\Gamma] + c_2c_0\kappa^c(\Gamma) - c_2(36\kappa^c(\Gamma) + 8(\delta')^{-1}[\Gamma]) \\ &= (c_3 - 8(\delta')^{-1}c_2)[\Gamma] + c_2(c_0 - 36)\kappa^c(\Gamma) \geq c_2(2[\Gamma] + 6\kappa^c(\Gamma)) \end{aligned}$$

since $c_3 \geq 9(\delta')^{-1}c_2$, and $c_0 \geq 42$.

Case (a) of the lemma is completely analogous. □

Lemma 10.4. *Let Δ be a regular comb of base widths at most $15N$, where the history H^Δ of Δ has no rules $(23)^{\pm 1}$. Assume that Δ has neither t - nor t' -cells and has no one-Step long subcombs Γ' such that $\text{Area}(\Gamma') \leq c_1[\Gamma'] + c_1\kappa^c(\Gamma')$. If Δ has a (12) -band \mathcal{T} with base of the form $\dots p_1 p_1^{-1} s_0^{-1} \dots$, then Δ has a long subcomb Γ with $\text{Area}(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma)$.*

Proof. Let us consider the maximal q -bands \mathcal{C} and \mathcal{C}_0 of Δ corresponding to the distinguished letters p_1 and s_0^{-1} , resp., in the base of \mathcal{T} . By Property (ix) for the trapezium $TP(\mathcal{C}, \mathcal{C}_0)$, the history H of \mathcal{C} has no subhistories of the form $(12)(2)(12)^{-1}$. It follows that H has type $(2)(1)(2)$ (or a subword of $(2)(1)(2)$), in particular, H has at most two rules $(12)^{\pm 1}$, and by (ix), all p_1 -cells corresponding to the rules of Step 2 are active from both sides in the subcomb Γ with handle \mathcal{C} as well. Below the usual notation $h, \mathbf{y}, \mathbf{z}, \mathbf{y}', \mathbf{z}'$ will be used for Γ .

We consider the system of derivative bands $\mathcal{C}_1, \dots, \mathcal{C}_s$ of Γ . Let them have histories H_1, \dots, H_s , respectively. Then $H_i \equiv H_i(1)H_i(2)H_i(3)$ ($i \leq s$), where $H_i(1)$ and $H_i(3)$ are of type (2) and $H_i(2)$ is of type (1), and some of the factors can be empty. The lengths of the histories are denoted by h_i and $h_i(j)$, $j = 1, 2, 3$. As usual, Γ_i is the derivative subcomb with handle \mathcal{C}_i .

Assume that \mathcal{C}_i is a derivative p_1^{-1} -band. Then H_i has neither rules $(12)^{\pm 1}$ nor rules of Step 1 by (v). Hence the history H_i is one-Step (of Step 2) and moreover all $p_1^{\pm 1}$ -bands of Γ_i are active from both sides and all other q -bands of Γ_i are passive by (vi). Therefore one can apply Lemma 7.18 (b) to the subcomb whose handle is a left-most maximal q -band of Γ_i which contradicts the assumption of the lemma about long subcombs. Hence, by (i), all the derivative bands are s_0 -bands. By the same argument and by Lemma 9.7, derivative subcombs cannot have one-Step histories. (Indeed, since the s_0 -band is passive by (ii), we have $|\mathbf{z}^{\Gamma_i}| - |\mathbf{y}^{\Gamma_i}| = |(\mathbf{z}')^{\Gamma_i}| - |(\mathbf{y}')^{\Gamma_i}| + 2$, and Lemma 9.7 would imply $\text{Area}(\Gamma_i) = \text{Area}'(\Gamma_i) + h^{\Gamma_i} \leq c_1[\Gamma_i] + c_1\kappa^c(\Gamma_i)$ contrary to the condition of the lemma.) Therefore each of the derivative bands crosses a (12) -band, and so $s \leq 2$.

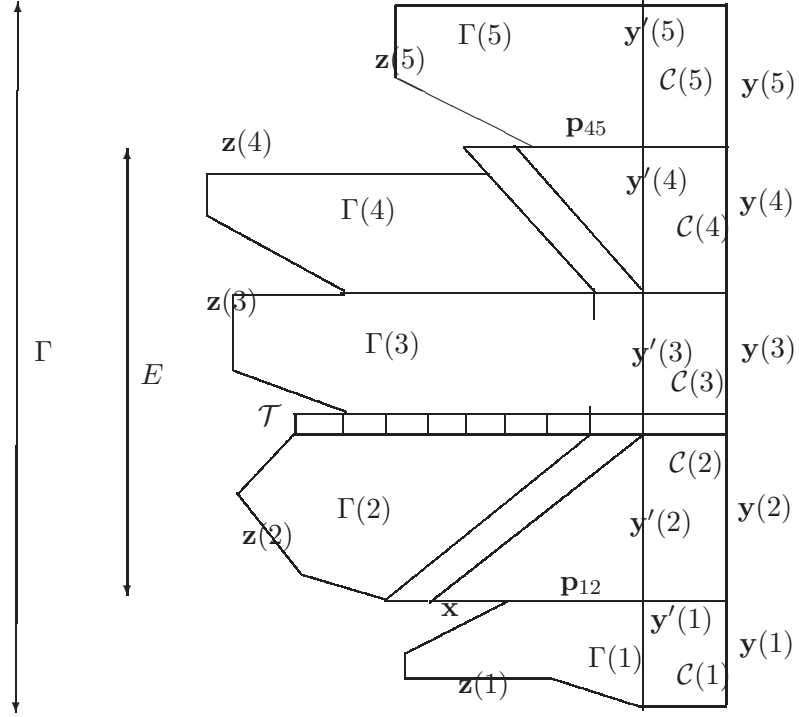
We factorize the history H as $H'H(3)H''$, where $H(3)$ is of Step 1 including the $(12)^{\pm 1}$ -rules, and H', H'' are of Step 2. If the H' -part (H'' -part) of Γ is non-empty and it is crossed by a derivative band of Γ , then this unique band has to cross the $(12)^{\pm 1}$ -band separating the $H(3)$ -part from the H' -part (from H'' -part). Using this observation, we further factorize $H' \equiv H(1)H(2)$ so that every maximal θ -band of the $H(2)$ -part $\Gamma(2)$ of Γ crosses a derivative s_0 -band ($H(2)$ can be empty) and the $H(1)$ -part $\Gamma(1)$ has no derivative bands. Similarly, we have $H'' \equiv H(4)H(5)$.

Thus $H \equiv H(1) \dots H(5)$. Let $h(1), \dots, h(5)$ be corresponding lengths of the histories (some of them may be zero), $\Gamma(i)$ ($i = 1, \dots, 5$) be the corresponding parts of Γ (some of them may be empty) with handles $\mathcal{C}(i)$, and $\mathbf{z} = \mathbf{z}(5) \dots \mathbf{z}(1)$, where $\mathbf{z}(i)$ is the common part of \mathbf{z} and $\partial\Gamma(i)$. Note that if the maximal (12) -band \mathcal{T} separating $\Gamma(3)$ and $\Gamma(2)$ is not crossed by a derivative band, then by (v), \mathcal{T} has no cells except for the intersection cell with the handle, and so the boundaries of $\Gamma(3)$ and $\Gamma(2)$ have no common edges except for that in \mathcal{C} . Similar note is true for the combs $\Gamma(3)$ and $\Gamma(4)$.

Let \mathbf{x} be the extension of the path $\mathbf{z}(1)^{-1}$ along z^{-1} , such that $|\mathbf{x}|_\theta = |\mathbf{z}(1)|_\theta$ and the last edge of \mathbf{x} is the s_0^{-1} -edge of $\mathbf{z}(2)$. Then every maximal a -band starting on $\mathbf{y}'(2)\mathbf{y}'(1)$ must end on \mathbf{x} by Lemma 5.6 because (a) the derivative s_0 -band is passive, (b) it has a common (12) -edges with \mathcal{C} by (v), and (c) every maximal a -band of $\Gamma(2)$ starting on the side $\mathbf{y}'(2)\mathbf{y}'(1)$ of $\mathcal{C}(1)\mathcal{C}(2)$ cannot end on $\mathbf{y}'(2)\mathbf{y}'(1)$ by (vii). Since all cells of $\mathcal{C}(1)\mathcal{C}(2)$ are active, the path \mathbf{x} has at least $h(1) + h(2)$ a -edges and only $h(1)$ θ -edges. It follows

therefore from Lemma 5.20 (a) that

$$|\mathbf{x}| - h(1) \geq 1 + \delta h(2) + \delta' h(1) \quad (10.38)$$



(Here 1 is added for the q -edge of the handle \mathcal{C} .) Since the path $\mathbf{z}(2)$ has $h(2)$ θ -edges which do not belong to \mathbf{x} , we get from 10.38:

$$|\mathbf{z}(2)\mathbf{z}(1)| - h(1) - h(2) \geq 1 + \delta h(2) + \delta' h(1) \text{ and } |\mathbf{z}(5)\mathbf{z}(4)| - h(4) - h(5) \geq 1 + \delta h(4) + \delta' h(5) \quad (10.39)$$

The band $\mathcal{C}(3)$ is passive by (ii); therefore $\mathbf{y}(3) = h(3)$ and so $|\mathbf{y}| = |\mathbf{y}'| = h + \delta'(h(1) + h(2) + h(4) + h(5))$. This together with equality $|\mathbf{z}(i)| \geq |\mathbf{z}(i)|_\theta = h(i)$, Lemma 5.20 (b), and (10.39) give rise to inequality

$$\begin{aligned} |\mathbf{z}| - |\mathbf{y}| &\geq |\mathbf{z}(1)| + \dots + |\mathbf{z}(5)| - |\mathbf{y}(1)| - \dots - |\mathbf{y}(5)| - 4(\delta - \delta') \geq \\ &2 + (\delta - \delta')(h(2) + h(4) - 4) > \max(1, (\delta - \delta')(h(2) + h(4))) \end{aligned} \quad (10.40)$$

In particular, Γ is a long subcomb.

Let $\mathbf{p}_{1,2}$ be the common segment of $\partial\Gamma(2)$ and $\partial\Gamma(1)$; it is a top/bottom path of a θ -band, and it consists of a p_1 -edge and of a -edges. The path $\mathbf{p}_{4,5}$ is defined similarly. Since every maximal a -band of $\Gamma(2)$ starting on $\mathbf{p}_{1,2}$ must end on $\mathbf{y}'(2) = (\mathbf{y}')^{\Gamma(2)}$, we have

$$|\mathbf{p}_{1,2}|_a \leq |\mathbf{y}'(2)|_a = h(2) \text{ and also } |\mathbf{p}_{4,5}|_a \leq |\mathbf{y}'(4)|_a = h(4) \quad (10.41)$$

Now we set $E = \Gamma(2) \cup \Gamma(3) \cup \Gamma(4)$. Then $\mathbf{z}^E = \mathbf{p}_{1,2}\mathbf{z}(2)\mathbf{z}(3)\mathbf{z}(4)\mathbf{p}_{4,5}$ and

$$|\mathbf{z}^E| - |\mathbf{y}^E| \leq |\mathbf{z}| - |\mathbf{y}| + \delta(|\mathbf{p}_{1,2}|_a + |\mathbf{p}_{4,5}|_a) = |\mathbf{z}| - |\mathbf{y}| + \delta(h(2) + h(4)) \quad (10.42)$$

by (10.41), since the maximal θ - and a -bands starting on $\mathbf{y}'(1)$ and $\mathbf{y}'(5)$ end on $\mathbf{z}(1)$ and $\mathbf{z}(5)$, respectively.

The comb E has $s \leq 2$ derivative subcombs $E_j = \Gamma_j$ ($j \leq s \leq 2$) whose handles are \mathcal{C}_j -s. By Lemma 10.2,

$$\sum_j \text{Area}(E_j) \leq c_2 \left(\left(\sum_j |(\mathbf{z}')^{E_j}| - h^{E_j} \right) + h(2) + h(4) + 1 \right) (h(2) + h(3) + h(4)) + \sum_j \kappa^c(E_j) \quad (10.43)$$

Since $\sum_j (|(\mathbf{z}')^{E_j}| - |h^{E_j}|) \leq |\mathbf{z}'| - |\mathbf{z}(1)| - |\mathbf{z}(5)| - h(2) - h(3) - h(4)$ and $h(2) + h(3) + h(4) \leq h$, the Inequalities (10.40) and (10.43) imply

$$\sum_j \text{Area}(E_j) \leq c_2 h (|\mathbf{z}'| - |\mathbf{z}(1)| - |\mathbf{z}(5)| - h(2) - h(3) - h(4) + (\delta - \delta')^{-1} (|\mathbf{z}| - |\mathbf{y}|) + 1) + c_2 \sum_j \kappa^c(E_j) \quad (10.44)$$

Observe that any simple band of $\Gamma(3)$ has no cells except for one cell π of the handle $\mathcal{C}(3)$ because there are no (θ, a) -cells from the left of π by (v). Hence the comb E consists of the cells of E_j -s, the cells of the handle of E and the cells of maximal a -bands connecting this handle with $\mathbf{p}_{1,2}$ and $\mathbf{p}_{4,5}$ and intersecting at most $h(2)$ and $h(4)$ θ -bands, respectively, by Lemma 5.6. Therefore we obtain from (10.44), (10.41), and Lemma 7.9 (a) that

$$\begin{aligned} \text{Area}(E) &\leq c_2 h (|\mathbf{z}'| - |\mathbf{z}(1)| - |\mathbf{z}(5)| - h(2) - h(3) - h(4) + \\ &\quad (\delta - \delta')^{-1} (|\mathbf{z}| - |\mathbf{y}|) + 1) + c_2 \kappa^c(E) + h + h(2)^2 + h(4)^2 \leq \\ c_2 h (|\mathbf{z}| - |\mathbf{z}(1)| - |\mathbf{z}(5)| - h(2) - h(3) - h(4) + (\delta - \delta')^{-1} (|\mathbf{z}| - |\mathbf{y}|)) &+ c_2 \kappa^c(E) + h(h(2) + h(4)) \end{aligned} \quad (10.45)$$

because $|\mathbf{z}| = |\mathbf{z}'| + 2$. Since $|\mathbf{y}(2)| + |\mathbf{y}(3)| + |\mathbf{y}(4)| = h(2) + h(3) + h(4) + \delta' (h(2) + h(4)) \leq h(2) + h(3) + h(4) + \delta' (\delta - \delta')^{-1} (|\mathbf{z}| - |\mathbf{y}|)$ by (10.40), Inequality (10.45) yields

$$\begin{aligned} \text{Area}(E) &\leq c_2 h (|\mathbf{z}| - |\mathbf{z}(1)| - |\mathbf{z}(5)| - |\mathbf{y}(2)| - |\mathbf{y}(3)| - |\mathbf{y}(4)| + \\ &\quad + (\delta' + 1) (\delta - \delta')^{-1} (|\mathbf{z}| - |\mathbf{y}|)) + c_2 \kappa^c(E) + h(h(2) + h(4)) \end{aligned} \quad (10.46)$$

Since the combs $\Gamma(1)$ and $\Gamma(5)$ have no derivative bands, applying Lemma 7.18 (b), we obtain the upper estimates

$$\text{Area}(\Gamma(1)) \leq (\delta')^{-1} h (|\mathbf{z}(1)| + |\mathbf{p}_{1,2}| - |\mathbf{y}(1)|) \leq c_2 h (|\mathbf{z}(1)| + 1 + \delta h(2) - |\mathbf{y}(1)|) \quad (10.47)$$

as $|\mathbf{p}_{1,2}| \leq 1 + \delta h(2)$ by (10.41) and Lemma 5.20; and similarly,

$$\text{Area}(\Gamma(5)) \leq c_2 h (|\mathbf{z}(5)| + 1 + \delta h(4) - |\mathbf{y}(5)|) \quad (10.48)$$

Summing Inequalities (10.46)-(10.48) and using (10.40), we get

$$\begin{aligned} \text{Area}(\Gamma) &\leq c_2 h (|\mathbf{z}| - |\mathbf{y}|) (2 + (\delta' + 1 + \delta) (\delta - \delta')^{-1}) + c_2 \kappa^c(E) \leq \\ &2\delta^{-1} c_2 [\Gamma] + c_2 \kappa^c(E) \end{aligned} \quad (10.49)$$

The handle \mathcal{C} has (at most) two (12)-cells (see the first paragraph in the proof of the lemma), and so it has at most 3 maximal subbands without (12)-cells. It follows from the definition of the λ -mixture that

$$\lambda(\mathbf{y}) \leq (h(1) + h(2))(h(3) + h(4) + h(5)) + (h(4) + h(5))h(3) \quad (10.50)$$

If $h(1) + h(5) \geq h(2) + h(4)$, then $2h(1) + 2h(5) \geq (h(1) + h(2)) + (h(4) + h(5))$, and so the right-hand side of (10.50) does not exceed $2h(1)(h - h(1)) + 2h(5)(h - h(5))$ which, in turn, does not exceed $4\kappa^c(\Gamma)$ since the ends of s_0 -bands separate the θ -edges of z in parts with $h(1), h(5)$, and $h - h(1) - h(5)$ θ -edges. If $h(1) + h(5) < h(2) + h(4)$, then the right-hand side of (10.50) does not exceed $2(h(2) + h(4))h \leq 2(\delta - \delta')^{-1}h(|\mathbf{z}| - |\mathbf{y}|)$ by (10.40). Thus, in any case $\lambda^c(\Gamma) \geq -\lambda(\mathbf{y}) \geq -4\kappa^c(\Gamma) - 2(\delta - \delta')^{-1}[\Gamma]$. Therefore by this inequality and (10.49),

$$\begin{aligned} \text{Area}(\Gamma) &\leq 2\delta^{-1}c_2[\Gamma] + c_2\kappa^c(\Gamma) \leq (c_3 - 2(\delta - \delta')^{-1}c_2)[\Gamma] + c_2(c_0 - 4)\kappa^c(\Gamma) = \\ &c_3[\Gamma] + c_2(c_0\kappa^c(\Gamma) - 4\kappa^c(\Gamma) - 2(\delta - \delta')^{-1}[\Gamma]) \leq c_3[\Gamma] + c_2(c_0\kappa^c(\Gamma) + \lambda^c(\Gamma)) = c_3[\Gamma] + c_2\mu^c(\Gamma) \end{aligned}$$

since $c_0 \geq 5$ and $c_3 > 5\delta^{-1}c_2$. The lemma is proved. \square

Lemma 10.5. *Let Γ be a regular comb of base width $b \leq 15N$. Assume that it has no maximal $t^{\pm 1}$ -, $(t')^{\pm 1}$ -bands except for the handle \mathcal{C} , and \mathcal{C} is either (a) a $(t')^{\pm 1}$ -band or (b) a $t^{\pm 1}$ -band. Also assume that there are no special θ -edges in any derivative subcomb. Let Γ in case (a), have $(23)^{\pm 1}$ -bands but have no $(12)^{\pm 1}$ -bands, and in case (b), it has $(12)^{\pm 1}$ -bands but has no $(23)^{\pm 1}$ -bands. Then $\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] + c_2\mu^c(\Gamma)$.*

Proof. We will consider the case (b) only. Observe that the (12) -cells of the $t^{\pm 1}$ -band \mathcal{C} are special, and so $\lambda(\mathbf{y}) = 0$. Let $\mathcal{C}_1, \dots, \mathcal{C}_s$ be the system of derivative bands of Γ . It follows from (i) and the assumptions of the lemma that all of them must be k^{-1} -bands.

First assume that a derivative band \mathcal{C}_i is of length $h_i = 1$. Then $\text{Area}(\Gamma_i) \leq 4(\delta')^{-1}[\Gamma_i]$ by Lemma 7.15. Since the statement of the lemma can by induction be assumed true for the comb $\Gamma \setminus \Gamma_i$, this implies that the statement is true for Γ since $\mu^c(\Gamma) \geq \mu^c(\Gamma \setminus \Gamma_i) \geq 0$ and $[\Gamma] = [\Gamma_i] + [\Gamma \setminus \Gamma_i]$. Thus we may assume further that $h_i > 1$ for every i .

Case 1. There is no derivative band whose length h_{i_0} is at least $0.8 \sum_{i=1}^s h_i$. Then $\sum h_i^2 \leq 4hh_- \leq 8\kappa^c(\Gamma_i)$ by Lemma 7.5. By Lemmas 7.13 and 7.10 (a), $\text{Area}(\Gamma_i) \leq 60Nh_i^2 + 2(\delta')^{-1}h_i(|\mathbf{z}_i| - |h_i|)$, and therefore by Lemma 7.9, $\sum \text{Area}(\Gamma_i) \leq 480N\kappa^c(\Gamma) + 2(\delta')^{-1}[\Gamma]$.

Case 2. There is a derivative band \mathcal{C}_{i_0} with $h_{i_0} \geq 0.8 \sum_{i=1}^s h_i$, and there is a short derivative \mathcal{C}'_j in it of length $h'_j > 0.6h' \geq 0.2 \sum h_i$. (Here we assume that the total some of lengths of short derivatives h' is at least $\sum h_i/3$ because $h_i > 1$ for every i .) Then, by (ii) and (vii), at least $h'_j/3$ a -bands starting on \mathcal{C}'_j must end on z . Therefore by Lemmas 7.13 and 7.10 (a),

$$\begin{aligned} \sum \text{Area}(\Gamma_i) &\leq \sum 60Nh_i^2 + 2(\delta')^{-1}[\Gamma] \\ &\leq 60Nh|\mathbf{z}|_a(0.2)^{-1}(1/3)^{-1} + 2(\delta')^{-1}[\Gamma] \leq 900N(\delta')^{-1}[\Gamma] + 2(\delta')^{-1}[\Gamma] \leq 0.5(\delta')^{-2}[\Gamma] \end{aligned}$$

since $(\delta')^{-1} \geq 2000N$.

Case 3. There is a derivative \mathcal{C}_{i_0} with $h_{i_0} \geq 0.8 \sum_{i=1}^s h_i$, and there are no short derivatives \mathcal{C}'_j of length $h'_j > 0.6h'$. Then $h'_- \geq h' - 0.6h' \geq 0.4 \sum h_i/3 = \frac{2}{15} \sum h_i$. Hence by Lemmas 7.13 and 7.6 ,

$$\sum \text{Area}(\Gamma_i) \leq 60Nh_i^2 + 2(\delta')^{-1}[\Gamma] \leq \frac{15}{2}60Nh h'_- + 2(\delta')^{-1}[\Gamma] \leq 2700N\lambda^c(\Gamma) + 2(\delta')^{-1}[\Gamma]$$

Thus, in any case

$$\sum \text{Area}(\Gamma_i) \leq 480N\kappa^c(\Gamma) + 2700N\lambda^c(\Gamma) + 0.5(\delta')^{-2}[\Gamma] \quad (10.51)$$

By Lemmas 7.12, 7.5, and 7.6, the number of cells in all simple bands of Γ does not exceed

$$h(h_- + h'_- + (\delta')^{-1}(|\mathbf{z}| - h - 1) \leq 2\kappa^c(\Gamma) + 6\lambda^c(\Gamma) + (\delta')^{-1}[\Gamma] \quad (10.52)$$

The two upper bounds (10.51) and (10.52) together prove the lemma since $c_0 \geq 1$ and $c_2 > 2800N$. \square

Lemma 10.6. *Let Γ be a regular comb of base width b , where $3N \leq b \leq 15N$. Assume that its history either (a) contains (12)-rules but does not contain (23)-rules or (b) vice versa. Then it admits a long quasicomb Δ such that $\text{Area}(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)$.*

Proof. Here we consider case (a) only. If Γ has a $(t')^{\pm 1}$ -band, the statement follows from Lemma 10.3 provided this band crosses a maximal (12)-band, and it follows from Lemma 9.2 (2) otherwise since λ takes non-negative values on one-Step combs and every subcomb with (passive) $(t')^{\pm 1}$ -handle is long. We may therefore assume that Γ has no such bands.

Assume Γ has a left-most maximal $t^{\pm 1}$ -band \mathcal{B} . If \mathcal{B} does not have (12)-cells, then again the statement follows from Lemma 9.2 (2). So we assume further that \mathcal{B} has (12)-cells.

Let Δ be the subcomb with the handle \mathcal{B} . If there are no special θ -edges from the left of each derivative band of Δ , then the statement follows from Lemma 10.5 since $c_3 > (\delta')^{-2}$. Therefore we may assume that Δ has a maximal (12)-band \mathcal{T} crossing a derivative band \mathcal{B}_i of \mathcal{B} and having a special θ -edge from the left of \mathcal{B}_i .

Note that \mathcal{B}_i is a k^{-1} -band by the choice of \mathcal{B} and by (i). Since only t -bands and $k^{\pm 1}$ -bands can have special (12)-edges (and k^{-1} -band can have them from the right only), it follows that \mathcal{T} has a k -cell from the left of \mathcal{B}_i . Since Γ has no (t') -cells, the base of \mathcal{T} is not aligned between the letter k and the next letter k^{-1} by (i). Moreover it has a subword $p_1p_1^{-1}s_0^{-1}$ between the k - and k^{-1} -letters by (v). So the statement of the lemma follows from Lemma 10.4. (Again, we take into account that λ is non-negative on one-Step combs.) Thus we may further assume that Γ has no t -cells.

By Lemma 9.6, one may assume that Γ has no one-Step subcombs of base width $> 2N$. Since $b \geq 3N$, this implies the existence of a (12)-band \mathcal{T} with base B_0 of length $\geq N$. Since $N > 2||B||$ and B_0 has neither t - nor t' -letters, it must have at least two subwords of the form $q^{\pm 1}q^{\mp 1}$ for some base letter q (see (i)). But the existence of $s_1^{-1}s_1$ excludes the possibility of all other subwords $q^{\pm 1}q^{\mp 1}$ by (v), and also by (v), the existence of $k^{-1}k$ implies the existence of at least one subword $s_0p_1p_1^{-1}s_0^{-1}$. Thus in any case B must have a subword $p_1p_1^{-1}s_0^{-1}$, which finishes the proof as in the previous paragraph. \square

11 Combs with multi-Step histories

In this section, we allow all three Steps in comb histories. Although Lemma 11.8 gives no estimate of the area if the size of a comb is close, in a sense, to one of the numbers T_i -s, this lemma (together with the lemmas of the next section) will imply that the Dehn functions of the groups M and G are almost quadratic because the set of T_i -s has infinitely many very long gaps. Again, to obtain upper estimates of areas for various combs one should apply a skillful combination of a number of quadratic parameter. For example, Lemma 11.3 (and also Lemma 12.9 in the next Section) shows the use of the ν -mixture.

Let H be the history of a comb Γ . Consider a factorization $H \equiv H(1) \dots H(m)$, where no two non-empty factors are separated by empty ones. We say that this factorization is *firm* if for every $i = 1, \dots, m-1$,

- (a) for non-empty H_i and H_{i+1} , the last letter of H_i and the first letter of H_{i+1} must belong to different Steps; so one of these two letters is $(12)^{\pm 1}$ - or $(23)^{\pm 1}$ -letter calling $(i) - (i+1)$ *transition letters*; the maximal θ -band of Γ corresponding to the $(i) - (i+1)$ transition letter is an $(i) - (i+1)$ -*transition band*;
- (b) the transition θ -bands of Γ are not simple.

There might be many firm factorizations of Γ . Observe that if a factorization $H \equiv H(1) \dots H(m)$ is firm, then $H^{-1} \equiv H(m)^{-1} \dots H(1)^{-1}$ is a firm factorization for the history of the mirror copy Γ^{-1} of the comb Γ .

Lemma 11.1. *Let Γ be a comb of base width $b \leq 15N$ with a firm factorization of the history $H \equiv H(1)H(2)H(3)$, where $H(2)$, $H(3)$ are one-Step histories, and $h(2) \geq 3h(3)$ (or $h(3) \geq 3h(2)$). Assume that the handle \mathcal{C} of Γ is a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band, and the $(1) - (2)$ transition band has no (θ, a) -cells between \mathcal{C} and the derivative band crossing this transition band, and the $H(2)$ -part (resp., the $H(3)$ -part) of Γ has passive k - or k' -cells only in the (12) - or in the (23) - bands. Then provided $h(2) \geq 0.01h$ (respectively, $h(3) \geq 0.01h$), there is a long subcomb Δ in Γ with $\text{Area}(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)$.*

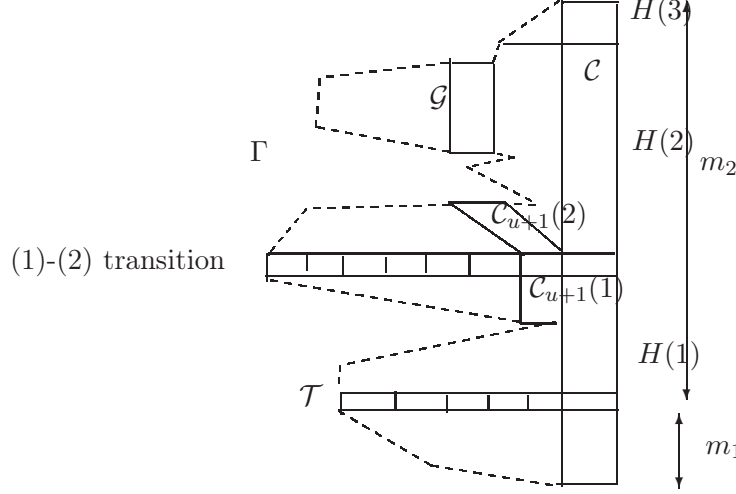
Proof. We will prove the lemma assuming that $h(2) \geq 3h(3)$ and $h(2) \geq 0.01h$ (> 0) since the proof of the second version of the lemma is similar.

Consider the system of derivative bands $\mathcal{C}_1, \dots, \mathcal{C}_s$ of Γ . Let $\mathcal{C}_1, \dots, \mathcal{C}_u$ have histories H_1, \dots, H_u which are subwords of $H(1)$, \mathcal{C}_{u+1} have history $H_{u+1} \equiv H_{u+1}(1)H_{u+1}(2)$, where $H_{u+1}(1)$ and $H_{u+1}(2)$ are a suffix and a prefix of $H(1)$ and $H(2)$, respectively: \mathcal{C}_{u+1} is a union of bands $\mathcal{C}_{u+1}(1)$ and $\mathcal{C}_{u+1}(2)$ having these two histories, and $h_{u+1} = h_{u+1}(1) + h_{u+1}(2) > 0$. Similarly we define subwords H_{u+2}, \dots, H_v of $H(2)$ and $H_{v+1} \equiv H_{v+1}(2)H_{v+1}(3)$, while H_{v+2}, \dots, H_s are subword of $H(3)$. (It is also possible that \mathcal{C}_{u+1} has history $H_{u+1}(1)H_{u+1}(2)H_{u+1}(3)$, where $H_{u+1}(2) \equiv H(2)$, and we will come back to this case later on.)

Proving by contradiction, we assume that Γ has no subcombs Δ with area satisfying the statement of the lemma.

The band $\mathcal{C}_{u+1}(2)$, if it has non-zero length, is not a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band by the condition on the $(1) - (2)$ -transition band and by (v). If some \mathcal{C}_i is a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band, for $i \in [u+2, v+1]$, then the derivative subcomb Γ_i satisfies the condition of Lemma 9.2, a contradiction since $c_3 > c_1$. Therefore all the derivative bands of the system $\mathcal{C}_{u+1}(2), \mathcal{C}_{u+2}, \dots, \mathcal{C}_{v+1}(2)$ are k^{-1} - or k' -bands by (v).

Assume that one of the numbers $h_{u+1}(2), h_{u+2}, \dots, h_{v+1}(2)$, redenote it by g , is at least $0.9h(2) \geq 0.009h > \delta h$



Denote by \mathcal{G} the corresponding derivative (sub)band of length g . Since $h(2) \geq 3h(3)$, we see that at most $h(2)/3 + 0.1h(2) \leq h(2)/2 < 2g/3$ maximal a -bands starting on \mathcal{G} end on the other derivative bands of Γ . So we may apply Lemma 7.17 (a) to \mathcal{G} and get inequality

$$\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] \quad (11.53)$$

Now assume that each of $h_{u+1}(2), h_{u+2}(2), \dots, h_{v+1}(2)$ is less than $0.9h(2)$. It follows from this assumption that $\max_{i=1}^s h_i < h - h(2) + 0.9h(2) \leq (1 - 0.001)h$ since $h(2) \geq 0.01h$. Therefore, by Lemma 7.4, $l_- \geq \min(\sum_{i=1}^s h_i, h - \max_{i=1}^s h_i) \geq 0.001 \sum_{i=1}^s h_i$. Hence by Lemmas 7.13, 7.10 (1), and by inequality $\delta'^{-1} > \max(40N, 4000)$, we have

$$\begin{aligned} \sum_{i=1}^s \text{Area}(\Gamma_i) &\leq 60N \sum_{i=1}^s h_i^2 + \sum_{i=1}^s 2\alpha_i h_i \leq \\ 60Nh \times 1000l_- + 2h((\delta')^{-1}(|\mathbf{z}| - h) &\leq 2(\delta')^{-1}[\Gamma] + (\delta')^{-2}hl_-, \end{aligned} \quad (11.54)$$

where $\alpha_i = |\mathbf{z}^{\Gamma_i}|_a$.

By the inequalities (7.11) and $l_- \geq 0.001 \sum_{i=1}^s h_i$, the number n_s of cells in all the simple θ -bands of Γ satisfies inequality

$$n_s \leq h\left(\frac{3}{2} \sum_{i=1}^s h_i + (\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}|)\right) \leq h(1500l_- + (\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}|))$$

From this inequality, (11.54), and by Lemma 7.5, we have

$$\text{Area}(\Gamma) \leq c_1[\Gamma] + c_1hl_-/2 \leq c_1([\Gamma] + \kappa^c(\Gamma)) \quad (11.55)$$

since $c_1 \geq 3(\delta')^{-2}$.

If $\lambda^c(\Gamma) \geq 0$, the statement of the lemma follows from Inequalities (11.53) and (11.55). Then we will assume that $-\lambda(\mathbf{y}) \leq \lambda^c(\Gamma) < 0$.

To estimate $\lambda^c(\Gamma)$ from below, we again start with the assumption that $g \geq 0.9h(2)$, and so at least $g/2$ maximal a -bands end on \mathbf{z} . We have $|\mathbf{z}|_a \geq g/2 \geq 0.4h(2) \geq 0.004h$. Hence $|\mathbf{z}| - h \geq \delta'h/250$ by Lemma 7.10 (a). Then by Lemma 6.2 (a),

$$\lambda^c(\Gamma) \geq -\lambda(\mathbf{y}) \geq -h^2 \geq -250(\delta')^{-1}[\Gamma]$$

Since $c_1 \geq 2(\delta')^{-2}$, this estimate together with (11.53) yields $\text{Area}(\Gamma) \leq c_1[\Gamma] + \lambda^c(\Gamma) \leq c_1[\Gamma] + \mu^c(\Gamma)$. Here, the right-hand side does not exceed $c_3[\Gamma] + c_2\mu^c(\Gamma)$ because $c_3 > c_1c_2$, and so the lemma proved in case $g \geq 0.9h(2)$.

Now let $g < 0.9h(2)$. Since $\lambda(\mathbf{y}) > 0$, by Lemma 6.2(e), there is a maximal (12)- or (23)-band \mathcal{T} such that there are m_1 θ -bands crossing \mathcal{C} below \mathcal{T} , m_2 θ -bands crossing \mathcal{C} above \mathcal{T} , and $\lambda(\mathbf{y}) \leq 2m_1m_2$.

Assume first that \mathcal{T} belongs to the $H(1)$ -part of Γ . Then one of the two ends of the derivative band \mathcal{C}_{u+1} lies above \mathcal{T} but there are at least $0.1h(2)$ θ -bands above this end since $g \leq 0.9h(2)$. Therefore by Lemma 6.2 (d),

$$\kappa^c(\Gamma) \geq 0.1m_1h(2) \geq 0.001m_1h \geq 0.001m_1m_2 \geq \lambda(\mathbf{y})/2000.$$

Then assume that \mathcal{T} belongs to the $H(2)H(3)$ -part of Γ . Then $m_2 \leq h(3) \leq h(2)/3$, since the one step history $H(2)$ has no $(12)^{\pm 1}$ - or $(23)^{\pm 1}$ -rules. On the other hand, there is an (lower) end of the derivative \mathcal{C}_{v+1} such that there are at least $h(3)$ maximal θ -bands above it and at least $0.1h(2)$ below it since $g \leq 0.9h(2)$. Hence

$$\kappa^c(\Gamma) \geq 0.1h(3)h(2) \geq 0.1m_2(0.01h) \geq 0.001m_1m_2 \geq \lambda(\mathbf{y})/2000.$$

Thus $\lambda^c(\Gamma) \leq 2000\kappa(\Gamma)$ if $g \leq 0.9h(2)$. This inequality and (11.55) imply $\text{Area}(\Gamma) \leq c_1([\Gamma] + \mu^c(\Gamma))$ because $c_0 \geq 2001$. This leads to a contradiction since $c_3 > c_2 > c_1$.

The case where \mathcal{C}_{u+1} had history $H_{u+1}(1)H_{u+1}(2)H_{u+1}(3)$ with $H_{u+1}(2) \equiv H(2)$, can be treated as the above subcase with $g \geq 0.9h(2)$, since now $g = h(2)$. \square

We omit the proof of the following lemma since the argument would be just a simplified version of the proof given above for Lemma 11.1: instead of the inequalities $h(3) \geq 0.01h$ and $h(3) \geq 3h(2)$, below we have that $h'' \geq 0.7h$ (and so $h'' \geq \frac{7}{3}h'$) and $H(1)$ is empty.

Lemma 11.2. . Let Γ be a comb of basic width $b \leq 15N$ with a $t^{\pm 1}$ - or $(t')^{\pm 1}$ -handle \mathcal{C} , and let the history of Γ have a firm factorization $H \equiv H'H''$, where $h'' \geq 0.7h$ and H'' is of Step (2). Let the derivative bands \mathcal{C}_i be all a $k^{\pm 1}$ - or $(k')^{\pm 1}$ -bands. Then Γ has a long subcomb Δ with $\text{Area}(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)$.

\square

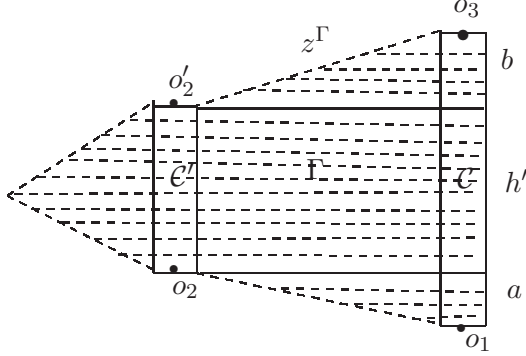
Lemma 11.3. Let Γ' be a subcomb of a comb Γ of base width $b \leq 15N$, \mathcal{C}' and \mathcal{C} their handles with histories H' and H , respectively, and each of these handles a $t^{\pm 1}$ - or a $(t')^{\pm 1}$ -band. Assume that $h' < h/2$, and H has at most 6 letters $(12)^{\pm 1}$ and $(23)^{\pm 1}$. Then either Γ has a long subcomb Δ with $\text{Area}(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)$ or $\text{Area}(\Gamma') \leq c_1([\Gamma'] + \lambda^c(\Gamma') + \nu_J^c(\Gamma) - \nu_J^c(\Delta'))$, where $\Delta' = \Gamma \setminus \Gamma'$.

Proof. If there is a maximal t - or t' -band in Γ , having no (12)- or (23)-cells, then by Lemma 9.2 (b), it is a handle of a long subcomb Δ with

$$\text{Area}(\Delta) \leq c_1[\Delta] + c_1\kappa^c(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)$$

because $\lambda^c(\Delta) \geq -\lambda(\mathbf{y}^\Delta) = 0$ for a one-Step Δ , and any subcomb with passive from the right handle is long.

Therefore we may assume that every t - or t' -band of Γ intersects a θ -band corresponding to one of the θ -letters $(12)^{\pm 1}$, $(23)^{\pm 1}$. Since their base widths are at most $15N$, the number of maximal t - and t' -bands in Δ does not exceed $90N < J/2$.



Now we will prove that

$$\nu_J^c(\Delta') < \nu_J^c(\Gamma) - (h')^2 \quad (11.56)$$

Recall that $\nu_J^c(\Gamma) = \nu_J(\mathbf{z}^\Gamma)$ by Lemma 7.2 (b). So we consider the two-colored string of beads responsible for the ν_J -mixture of \mathbf{z}^Γ . Denote by o_1 and o_3 the black beads on the two ends of \mathcal{C} and by o_2, o_2' the black beads on the two ends of \mathcal{C}' . We have h' white beads between o_2 and o_2' , a white beads between o_1 and o_2 and b white beads between o_2' and o_3 for some $a, b \geq 0$. Thus, $a + h' + b = h$. When we pass from \mathbf{z}^Γ to $\mathbf{z}^{\Delta'}$, we delete at least two black beads o_2, o_2' . But the number of black beads between the vertices o_1, o_3 is less than J . Hence we may apply Lemma 6.2, parts (d,c), and obtain that $\nu_J^c(\Delta') \leq \nu_J^c(\Gamma) - a(h' + b) - b(h' + a)$. But here $a(h' + b) + b(h' + a) > (h')^2$ since $a + b > h'$. So the inequality 11.56 is obtained.

Now by Lemmas 7.13 and 7.10,

$$\text{Area}(\Gamma') \leq 60N(h')^2 + 2(\delta')^{-1}[\Gamma'] \quad (11.57)$$

Since by Lemma 6.2(a) and (11.56),

$$\lambda^c(\Gamma') \geq -\lambda^c(\mathbf{y}^{\Gamma'}) > -(h')^2/2 \geq (-\nu_J^c(\Gamma) + \nu_J^c(\Delta'))/2 \quad (11.58)$$

we deduce from (11.57) and (11.58) that

$$\begin{aligned} \text{Area}(\Gamma') &\leq (60N + c_1/2 - c_1/2)(\nu_J^c(\Gamma) - \nu_J^c(\Delta')) + 2(\delta')^{-1}[\Gamma'] \\ &= (60N + \frac{c_1}{2})(\nu_J^c(\Gamma) - \nu_J^c(\Delta')) - \frac{c_1}{2}(\nu_J^c(\Gamma) - \nu_J^c(\Delta')) + 2(\delta')^{-1}[\Gamma'] \leq \\ &(60N + \frac{c_1}{2})(\nu_J^c(\Gamma) - \nu_J^c(\Delta')) + c_1\lambda^c(\Gamma') + 2(\delta')^{-1}[\Gamma'] \leq c_1(\nu_J^c(\Gamma) - \nu_J^c(\Delta') + \lambda^c(\Gamma') + [\Gamma']), \end{aligned}$$

because $c_1 \geq 61N + c_1/2$ and $c_1 \geq 2(\delta')^{-1}$, and the lemma is proved. \square

Lemma 11.4. *Let Δ be a comb with history H^Δ of type (1)(12)(2)(23)(3), where the (1)-part and the (3)-part of Δ can be empty. Assume that the base width b of Δ satisfies inequalities $4N < b \leq 15N$. Then either (a) Γ admits a long quasicomb with*

$$\text{Area}(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma) + c_3(\nu_J^c(\Delta) - \nu_J^c(\Delta \setminus \Gamma)) \quad \text{or}, \quad (11.59)$$

(b) Δ has a maximal $t^{\pm 1}$ or $(t')^{\pm 1}$ -band of length l , where $T_i \leq l < 10T_i$ for some i .

Proof. Δ has a regular subcomb Δ_1 of base widths $b_1 > 3N$ such that the base widths of the trapezium $\mathbf{T} = Tp(\mathcal{D}_1, \mathcal{D})$ is at least $N + 1$, where \mathcal{D}_1 and \mathcal{D} are the handles of Δ_1 and Δ , respectively. If the history of Δ_1 has one Step, then the Property (a) of the lemma is a consequence of Lemma 9.6 since in this case $\lambda^c(\Delta) \geq -\lambda(\mathbf{y}^\Delta) = 0$, and $\nu_J^c(\Delta) - \nu_J^c(\Delta \setminus \Gamma) \geq 0$ by Lemmas 7.2 and 7.3 (d). It is a consequence of Lemma 10.6 if the history of Δ_1 has no $(12)^{\pm 1}$ - or $(23)^{\pm 1}$ -rules. Hence we may assume that the history of Δ_1 is of type $(1)(12)(2)(23)(3)$ as well.

Since the history of \mathbf{T} has both rules $(12)^{\pm 1}$ and $(23)^{\pm 1}$, the base of \mathbf{T} is normal by (xi), and since the base of \mathbf{T} has at least $N + 1$ letters, \mathbf{T} contains a standard subtrapezium, and so the subtrapezium of \mathbf{T} bounded by the (12) - and (23) -bands has height T_i for some i .

Since \mathbf{T} has a normal base of length $\geq N + 1$, its base must contain a letter $t^{\pm 1}$ and a letter $(t')^{\pm 1}$. Denote by \mathcal{C} (\mathcal{C}') a maximal $t^{\pm 1}$ -band ($(t')^{\pm 1}$ -band) of Δ crossing \mathbf{T} . We may assume that neither of them corresponds to the first letter of the base of \mathbf{T} since otherwise this normal base of length $N + 1$ has one more $t^{\pm 1}$ or $(t')^{\pm 1}$ -letter, respectively, and one can select one of the bands $\mathcal{C}, \mathcal{C}'$ closer to \mathcal{D} .

By Γ and Γ' , we denote the subcombs with handles \mathcal{C} and \mathcal{C}' , respectively. Let the histories of these handles be H and H' . Without loss of the generality of our further proof, we assume that Γ' is contained in Γ . Since $0 \leq \nu_J^c(\Gamma) - \nu_J^c(\Gamma \setminus \Gamma') \leq \nu_J^c(\Delta) - \nu_J^c(\Delta \setminus \Gamma')$ by Lemmas 7.2 (b) and 7.3 (d,e), we may assume by Lemma 11.3, that $h' = h^{\Gamma'} \geq h/2$.

Let $H \equiv H(1)H(2)H(3)$ and $H' \equiv H'(1)H'(2)H'(3)$ be the Step factorizations. Since the left-most q -band of \mathbf{T} is not a subband of \mathcal{C}' or \mathcal{C} , the maximal (12) - and (23) -bands of Γ' and Γ are not simple, and so the factorizations $H \equiv H(1)H(2)H(3)$ and $H' \equiv H'(1)H'(2)H'(3)$ are firm. Recall that by (ii), every k -cell of the $H(1)$ - and $H(2)$ -parts of Γ (every k' -cell of the $H(2)$ - and $H(3)$ -parts of Γ) is not passive unless it belongs to a (12) - or (23) -band.

One may assume that $10h'(2) \leq h'$ because $h'(2) = T_i$ and if $10h'(2) > h'$ then the length of \mathcal{C}' belongs to the segment $[T_i, 10T_i)$, and we obtain Property (b). Similarly we may assume that $10h(2) \leq h$.

If $h(1) \geq 0.3h$, then $h(1) \geq 3h(2)$, and one can apply Lemma 11.1 to Γ^{-1} and obtain the desired estimate (11.59) for $\text{Area}(\Gamma)$. If $h(1) < 0.3h$, then $h'(1) \leq h(1) < 0.6h'$ since $h' \geq h/2$. It follows that $h'(3) = h' - h'(1) - h'(2) \geq (1 - 0.6 - 0.1)h' \geq 0.3h' \geq 3h'(2)$. Now one can apply Lemma 11.1 to Γ' and obtain the required estimate (11.59) for $\text{Area}(\Gamma')$. \square

Lemma 11.5. *Let Γ be a regular comb of width $b \leq 15N$ with a handle \mathcal{C} containing both (12) - and (23) -cells, and the history H of Γ contains, in its Step factorization, a product $H(1)H(2)H(3)$, where $h(2) \geq h/30$ and $h(1) + h(3) < h(2)/2$. Let one of the derivative bands \mathcal{C}_i be a $k^{\pm 1}$ - or $(k')^{\pm 1}$ -band which crosses all the maximal θ -bands of the $H(2)$ -part of Γ . Assume also that either*

(a) \mathcal{C} is a $t^{\pm 1}$ -band, and $H(1)H(2)H(3)$ is of the form $(2)(1)(2)$ or

(b) \mathcal{C} is a $(t')^{\pm 1}$ -band, and $H(1)H(2)H(3)$ is of the form $(2)(3)(2)$.

Then $\text{Area}(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma)$.

Proof. It follows from (xii) that H has no subwords of type $(1)(2)(1)$ or $(3)(2)(3)$ because Γ is regular and has both rules $(12)^{\pm 1}$ and $(23)^{\pm 1}$ in its history. An a -band starting on the $H(2)$ -part $\mathcal{C}_i(2)$ of \mathcal{C}_i cannot cross a (23) -band in case (a) or (12) -band in case (b) by (v). Also it cannot end on $\mathcal{C}_i(2)$ by (ii) and (vii). Hence every maximal a -band starting on $\mathcal{C}_i(2)$ must end either on the parts $\mathcal{C}_i(1)$, $\mathcal{C}_i(3)$ or on the path $\mathbf{z} = \mathbf{z}^\Gamma$. Now inequalities

$h(1) + h(3) < h(2)/2$ and $h(2) \geq h/30 \geq \delta h$ make possible to apply Lemma 7.17 to Γ . Hence

$$\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] \quad (11.60)$$

Since more than $\frac{1}{2}h(2) - 2 \geq \frac{1}{60}h - 2$ maximal a -bands end on \mathbf{z} , we have $|\mathbf{z}| - |\mathbf{y}| > \delta'(\frac{1}{60}h - 2) + 1 > \frac{\delta'}{60}h$ by Lemma 7.10 (a), i.e., $h < 60(\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}|)$. Hence by Lemma 6.2 (a),

$$\lambda^c(\Gamma) \geq -\lambda^c(\mathbf{y}) \geq -h^2/2 \geq -30(\delta')^{-1}h(|\mathbf{z}| - |\mathbf{y}|) = -30(\delta')^{-1}[\Gamma],$$

This inequality and (11.60) complete the proof of the lemma since

$$(\delta')^{-2}[\Gamma] = ((\delta')^{-2} + 30(\delta')^{-1}c_2)[\Gamma] - 30(\delta')^{-1}c_2[\Gamma] \leq c_3[\Gamma] + c_2\lambda^c(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma)$$

by the choice of c_3 , the definition of $\mu^c(\Gamma)$, and by Lemma 7.2 (a). \square

Lemma 11.6. *Let Γ be a regular comb of width $b \leq 15N$ whose handle is a $t^{\pm 1}$ -band (or $(t')^{\pm 1}$ -band) with firm factorization of the history $H \equiv H(1) \dots H(5)$, where $H(2)$ and $H(4)$ are both of type (1) (or both of type (3), respectively), and $h(2) + h(4) \geq 0.9h$. Assume that $H(3)$ contains both $(12)^{\pm 1}$ and $(23)^{\pm 1}$, and one of the derivative bands \mathcal{C}_i crosses all the maximal (12) - and (23) -bands of Γ . Then Γ has a long sumcomb Δ with $\text{Area}(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)$.*

Proof. We will prove only the first version of the lemma. The history H_i of \mathcal{C}_i contains $H(3)$, and so it contains the rule $(23)^{\pm 1}$, and therefore the band \mathcal{C}_i cannot be a $t^{\pm 1}$ -band; it is k^{-1} -band by (i). It follows from the assumption of the lemma that the derivative \mathcal{C}_i must cross $H(2)$ -, $H(3)$ - and $H(4)$ -parts of the comb Γ , and therefore $h_i > 0.9h$. Moreover the sum of lengths of $H(2)$ - and $H(4)$ -parts of \mathcal{C}_i is at least $0.9h$, and so $\max(h(2), h(4)) \geq 0.4h$.

There are at most $0.1h$ maximal a -bands starting on \mathcal{C}_i and ending on other derivative bands. There are no a -bands starting on the $H(2)$ -part and ending on the $H(4)$ -part of \mathcal{C}_i by the condition on $H(3)$, because an a -band cannot cross both (12) - and (23) -bands by (v). Besides both the $H(2)$ - and the $H(4)$ -part of \mathcal{C}_i are active from the right by (ii). Therefore we can apply Lemma 7.17 to Γ :

$$\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] \quad (11.61)$$

Note that at least $0.9h - 4 - 0.1h = 0.8h - 4$ maximal a -bands end on \mathbf{z} , and so $|\mathbf{z}| - |\mathbf{y}| - 2 = |\mathbf{z}'| - |\mathbf{y}'| \geq (0.8h - 4)\delta'$ by Lemma 7.10 (1), whence

$$h \leq \frac{5}{4}(\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}|) \quad (11.62)$$

Hence by Lemma 6.2 (a) and Inequality (11.62), we get

$$\lambda^c(\Gamma) \geq -\lambda(\mathbf{y}) > -h^2/2 > -h(\delta')^{-1}(|\mathbf{z}| - |\mathbf{y}|) = -(\delta')^{-1}[\Gamma]$$

This inequality and (11.61) complete the proof as in Lemma 11.5 because

$$(\delta')^{-2}[\Gamma] = ((\delta')^{-2} + (\delta')^{-1}c_2)[\Gamma] - (\delta')^{-1}c_2[\Gamma] \leq c_3[\Gamma] + c_2\lambda^c(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma)$$

\square

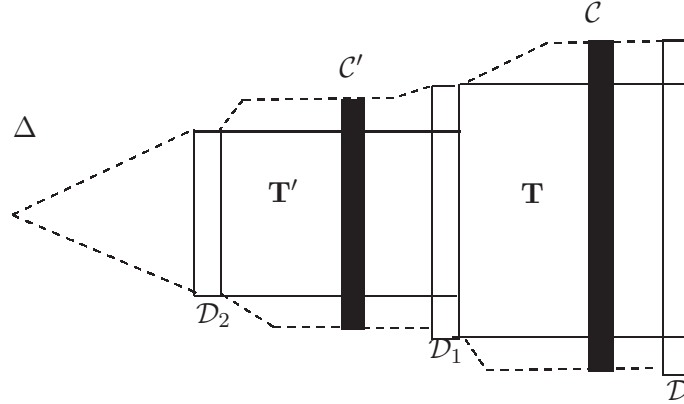
Lemma 11.7. *Let Δ be a regular comb. Assume that history H^Δ has $m \leq 6$ letters $(12)^{\pm 1}$ and $(23)^{\pm 1}$, and $\max(4, 2m)N < b \leq 15N$ for the base width b of Δ . Then either Δ admits a long quasicomb Δ' with*

$$\text{Area}(\Delta') \leq c_3[\Delta'] + c_2\mu^c(\Delta') + c_3(\nu_J^c(\Delta) - \nu_J^c(\Delta \setminus \Delta'))$$

or Δ has a maximal $t^{\pm 1}$ or $(t')^{\pm 1}$ -band of length l , where $T_i \leq l < 200T_i$ for some T_i .

Proof. If $m \leq 2$ or the handle of Δ does not contain either (12)-cells or (23)-cells, then the statement follows from Lemmas 9.6, 10.6, and 11.4 because λ^c is non-negative for one Step (quasi)combs, the third summand in the above inequality is positive by Lemmas 7.2 (a) and 7.3 (d), and $c_3 > c_2 > c_1$. Then we will induct on m assuming that $3 \leq m \leq 6$ and that the history H^Δ of Δ contains both rules $(12)^{\pm 1}$ and $(23)^{\pm 1}$.

The comb Δ has a regular subcomb Δ_1 of base width $b_1 > N(2m - 1)$ such that the base width of the filling trapezium $\mathbf{T} = Tp(\mathcal{D}_1, \mathcal{D})$ is $N + 1$, where \mathcal{D}_1 and \mathcal{D} are the handles of Δ_1 and Δ , respectively. If the history of Δ_1 has m_1 letters $(12)^{\pm 1}$ and $(23)^{\pm 1}$, and $m_1 < m$, then the statement of the lemma is a consequence of the inductive hypothesis since $2m_1 \leq 2m - 1$. Hence we may assume that $m_1 = m$. Similarly, Δ_1 has a regular subcomb Δ_2 of width $b_2 > (2m - 2)N$ with handle \mathcal{D}_2 and the filling trapezium $\mathbf{T}' = Tp(\mathcal{D}_2, \mathcal{D}_1)$ of width $N + 1$, and we may assume that $m_2 = m_1 = m$ since otherwise $2m_2 \leq 2m - 2$ and one may apply the inductive conjecture to Δ_2 .



Thus both \mathcal{D}_1 and \mathcal{D}_2 have (12)- and (23)-cells. Hence the base of \mathbf{T} is normal by (xi), and so it contains a letter $t^{\pm 1}$ and a letter $(t')^{\pm 1}$ which are not the first letter in this base. The same is true for \mathbf{T}' . Denote by \mathcal{C} (\mathcal{C}') a maximal $t^{\pm 1}$ -band or $(t')^{\pm 1}$ -band of Δ crossing \mathbf{T} (crossing \mathbf{T}') and corresponding to this letter of the base. By Γ and Γ' , we denote the subcombs with handles \mathcal{C} and \mathcal{C}' , respectively. The histories of these handles are H and H' . We will assume that $h' \geq h/2$ for their length, because otherwise one can apply Lemma 11.3 to Γ since $c_1 < c_2 < c_3$.

Observe that there are derivative bands in both Γ and Γ' crossing all the θ -bands of Γ corresponding to the rules $(12)^{\pm 1}$ and $(23)^{\pm 1}$. This follows from the equality of the numbers of (12)- and (23)-cells in \mathcal{D} and \mathcal{D}_2 . Hence such a derivative band is a k^{-1} or k' -band by (i) and (v), and there exist firm factorizations $H = H^{(1)} \dots H^{(m+1)}$ and $H' \equiv (H')^{(1)} \dots (H')^{(m+1)}$ for Γ and Γ' , where $(H')^i \equiv H^i$ for $i = 2, \dots, m - 1$.

Besides one may assume that all other derivative bands of Γ and Γ' (if any) are also either k^{-1} - or k' -bands. Indeed, they do not cross (12)- and (23)-bands and so if a derivative band is $t^{\pm 1}$ - or $(t')^{\pm 1}$ -band, one can apply Lemma 9.2 to a derivative diagram

Δ' , and the statement of our lemma follows. (Similarly, the comb Λ from Case 4 below, also enjoys this property of Γ and Γ' by the same reason.)

By Property (xii) the step history of Δ is a subword of $(2)(1)(2)(3)(2)(1)(2)$. Since $m \geq 3$ and one always can replace Δ by Δ^{-1} (and H by H^{-1}), we have to consider the following 6 step histories: $(1)(2)(3)(2)$, $(3)(2)(1)(2)$, $(1)(2)(3)(2)(1)$, $(2)(1)(2)(3)(2)$, $(1)(2)(1)(2)(3)(2)$, and $(2)(1)(2)(1)(2)(3)(2)$.

Case 1. The history H is of type $(1)(2)(3)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}$ is the corresponding firm factorization. In this case, we select \mathcal{C} to be a $t^{\pm 1}$ -band and \mathcal{C}' a $(t')^{\pm 1}$ -band.

If $h^{(4)} \geq 0.01h$, then one can apply Lemma 11.1 to Γ with $H(1) \equiv H^{(1)}H^{(2)}H^{(3)}$, $H(2) \equiv H^{(4)}$ and $H(3) \equiv \emptyset$, since the condition on the $(1) - (2)$ -transition holds by (v), and the passive cells of $H(2)$ -part of \mathcal{C} must be (12) - or (23) -cells by (ii). Hence we obtain a required subcomb. Therefore we may further assume that $h^{(4)} < 0.01h$.

Since the base of \mathbf{T} is normal and has $\geq N$ letters, this trapezium contains a standard subtrapezium with history $H^{(2)}$, and so $h^{(2)} = T_i$ for some i . We may assume that $T_i \leq h/200$ because otherwise $h < 200T_i$, and the lemma is true. Thus $h^{(2)} + h^{(4)} < h/60$.

Assume that $h^{(3)} \geq h/30$. Then $(h')^{(3)} = h^{(3)} \geq h'/30$ and $(h')^{(2)} + (h')^{(4)} \leq h^{(2)} + h^{(4)} < h/60 \leq h^{(3)}/2 = (h')^{(3)}/2$. Hence one can apply Lemma 11.5(b) to Γ' . (Here $H(1) \equiv (H')^{(2)}$, $H(2) \equiv (H')^{(3)}$ and $H(3) \equiv (H')^{(4)}$.) Therefore we can further assume that $h^{(3)} < h/30$.

Now, $h^{(1)} > h - h^{(2)} - h^{(3)} - h^{(4)} > h(1 - 1/60 - 1/30) = 0.95h$. Therefore Lemma 11.6 is applicable to Γ with $H(1) \equiv \emptyset$, $H(2) \equiv H^{(1)}$, $H(3) \equiv H^{(2)}H^{(3)}H^{(4)}$, $H(4) \equiv H(5) \equiv \emptyset$. This completes Case 1.

Case 2. The history H is of type $(3)(2)(1)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}$ is the corresponding firm factorization. In this case we will assume that \mathcal{C} is a $(t')^{\pm 1}$ -band and \mathcal{C}' a $t^{\pm 1}$ -band. Then the proof coincides with that in Case 1.

Case 3. The history H is of type $(1)(2)(3)(2)(1)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}$ is the corresponding firm factorization. In this case we will assume that \mathcal{C} is a $t^{\pm 1}$ -band.

As in Case 1, one may assume that $\max(h^{(2)}, h^{(4)}) \leq h/200$. Then $h^{(3)} < h/100$ by (xvi). Therefore $h^{(1)} + h^{(5)} > h - 2h/100 = 0.98h$. Therefore one can apply Lemma 11.6 to Γ with $H(1) = \emptyset$, $H(2) \equiv H^{(1)}$, $H(3) \equiv H^{(2)}H^{(3)}H^{(4)}$, $H(4) \equiv H^{(5)}$, and $H(5) \equiv \emptyset$.

Case 4. The history H is of type $(2)(1)(2)(3)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}$ is the corresponding firm factorization. In this case we will assume that both \mathcal{C} and \mathcal{C}' are $t^{\pm 1}$ -bands and consider an auxiliary maximal $(t')^{\pm 1}$ -band \mathcal{B} between them. It exists since the base of $Tp(\mathcal{C}', \mathcal{C})$ is normal, and determines a subcomb Λ of Δ whose history G has a firm factorization $G^{(1)} \dots G^{(5)}$.

If $h^{(5)} \geq 0.01h$, then one can apply Lemma 11.1 to Γ with $H(1) \equiv H^{(1)}H^{(2)}H^{(3)}H^{(4)}$, $H(2) = H^{(5)}$ and $H(3) = \emptyset$. Hence we may assume that $h^{(5)} < 0.01h$.

Since Δ is regular, $h^{(3)} = T_i$ for some i . We may assume that $T_i \leq h/200$ because otherwise $h < 200T_i$, as desired. Thus $h^{(3)} + h^{(5)} < h/60$.

Assume that $h^{(4)} \geq h/30$. Then $g^{(4)} = h^{(4)} \geq g/30$ and $g^{(3)} + g^{(5)} \leq h^{(3)} + h^{(5)} < h/60 \leq h^{(4)}/2 = g^{(4)}/2$. Hence one can apply Lemma 11.5 to Λ . (Here $H(1) \equiv G^{(3)}$, $H(2) \equiv G^{(4)}$ and $H(3) \equiv G^{(5)}$.) Therefore we can further assume that $h^{(4)} < h/30$.

Suppose $h^{(1)} \geq 0.7h$. Then Lemma 11.2 can be applied to Γ^{-1} with $H'' \equiv (H^{(1)})^{-1}$. Hence we may assume that $h^{(1)} < 0.7h$, and therefore $h^{(2)} > h(1 - 0.7 - 1/30 - 1/60) = h/4$.

If $g^{(1)} \geq 0.01g$, then Lemma 11.1 is applicable to Λ^{-1} with $H(2) \equiv (G^{(1)})^{-1}$ and $H(3) \equiv \emptyset$. Therefore we may assume that $g^{(1)} < 0.01g$.

Now $(h')^{(2)} = h^{(2)} > h/4 \geq h'/4$ and $(h')^{(1)} + (h')^{(3)} \leq g^{(1)} + h^{(3)} < 0.01g + h/200 \leq 0.015h < (h')^{(2)}/2$. Hence Lemma 11.5 is applicable to Γ' , and the lemma is proved in Case 4.

Case 5. The history H is of type $(1)(2)(3)(2)(1)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}H^{(6)}$ is the corresponding firm factorization. In this case we will assume that \mathcal{C} is a $(t')^{\pm 1}$ -band and \mathcal{C}' is a $t^{\pm 1}$ -band.

If $h^{(6)} \geq 0.01h$, then one can apply Lemma 11.1 to Γ with $H(1) \equiv H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}$, $H(2) \equiv H^{(6)}$ and $H(3) \equiv \emptyset$. Hence we may assume that $h^{(6)} < 0.01h$. Then, as in Case 3, we may assume that $\max(h^{(2)}, h^{(4)}) \leq h/200$ and $h^{(3)} < h/100$. Since $h' \geq h/2$, it follows that

$$\begin{aligned} (h')^{(1)} + (h')^{(5)} &= h' - (h')^{(2)} - (h')^{(3)} - (h')^{(4)} - (h')^{(6)} \geq \\ h' - h^{(2)} - h^{(3)} - h^{(4)} - h^{(6)} &> h' - 0.03h \geq h' - 0.06h' = 0.94h'. \end{aligned}$$

Therefore one can apply Lemma 11.6 to Γ' with $H(1) \equiv \emptyset$, $H(2) \equiv (H')^{(1)}$, $H(3) \equiv (H')^{(2)}(H')^{(3)}(H')^{(4)}$, $H(4) \equiv (H')^{(5)}$ and $H(5) \equiv (H')^{(6)}$.

Case 6. The history H is of type $(2)(1)(2)(3)(2)(1)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}H^{(6)}H^{(7)}$ is the corresponding firm factorization. In this case we will assume that \mathcal{C} is a $(t')^{\pm 1}$ -band and \mathcal{C}' is a $t^{\pm 1}$ -band.

If $h^{(7)} \geq 0.01h$, then one can apply Lemma 11.1 to Γ with $H(1) \equiv H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}H^{(6)}$, $H(2) \equiv H^{(7)}$ and $H(3) \equiv \emptyset$. Hence we may assume that $h^{(7)} < 0.01h$. Similarly, $h^{(1)} < 0.01h$. Then, as in Cases 3 and 5 we may assume that $\max(h^{(3)}, h^{(5)}) < h/200$ and $h^{(4)} < h/100$. Therefore

$$\begin{aligned} (h')^{(2)} + (h')^{(6)} &= h' - (h')^{(1)} - (h')^{(3)} - (h')^{(4)} - (h')^{(5)} - (h')^{(7)} \\ &\geq h' - h^{(1)} - h^{(3)} - h^{(4)} - h^{(5)} - h^{(7)} > h' - 0.04h \geq h' - 0.08h' = 0.92h'. \end{aligned}$$

Therefore one can apply Lemma 11.6 to Γ' with $H(1) \equiv (H')^{(1)}$, $H(2) \equiv (H')^{(2)}$, $H(3) \equiv (H')^{(3)}(H')^{(4)}(H')^{(5)}$, $H(4) \equiv (H')^{(6)}$ and $H(5) \equiv (H')^{(7)}$.

The lemma is proved in any case. \square

Lemma 11.8. *Let Δ be a comb of base width $b > 13N$. Then either Δ admits a long quasicomb Γ' with*

$$\text{Area}(\Gamma') \leq c_3[\Gamma'] + c_2\mu^c(\Gamma') + c_3(\nu_J^c(\Delta) - \nu_J^c(\Delta \setminus \Gamma')) \quad (11.63)$$

or Δ has a maximal $t^{\pm 1}$ or $(t')^{\pm 1}$ -band of length l , where $T_i \leq l < 200T_i$ for some T_i , and this band is a handle of a subcomb of base width $\leq 14N$.

Proof. Recall that the third term in the right-hand side of (11.63) is positive for every sub(quasi)comb Γ' by Lemma 7.3 (e) and Remark 9.5.

Then we observe that Δ has a subcomb Δ_0 of base with $b^{\Delta_0} \in (13N, 15N]$, and in turn, Δ_0 has a regular subcomb Γ with $12N < b^\Gamma \leq 14N$. If Γ is a one Step comb, then by Lemma 9.6, it admits a long quasicomb Γ' with $\text{Area}(\Gamma') \leq c_1([\Gamma'] + \kappa^c(\Gamma'))$. Here the right-hand side does not exceed $c_3[\Gamma'] + c_2\mu^c(\Gamma')$ since $c_1 < c_2 < c_3$ and $\lambda^c(\Gamma') \geq -\lambda(y^{\Gamma'}) = 0$ for one Step comb Γ' . Inequality (11.63) follows in this case.

Then we may assume that the history H of Γ has one of the rules $(12)^{\pm 1}$, $(23)^{\pm 1}$. If H has no $(12)^{\pm 1}$ or no $(23)^{\pm 1}$, then the statement of the lemma follows from Lemma 10.6. Otherwise H has at most 6 letters $(12)^{\pm 1}$ and $(23)^{\pm 1}$ by Properties (xii) and (viii), since Γ is a regular comb (and so there exists a trapezium of width $\geq N$ with history H). Now the application of Lemmas 11.7 and 7.3 (e) completes the proof. \square

12 Separation of a hub

In this section we consider minimal diagrams over the group G with cyclically reduced boundary paths. Thus in contrast to previous sections, we study diagrams with hubs.

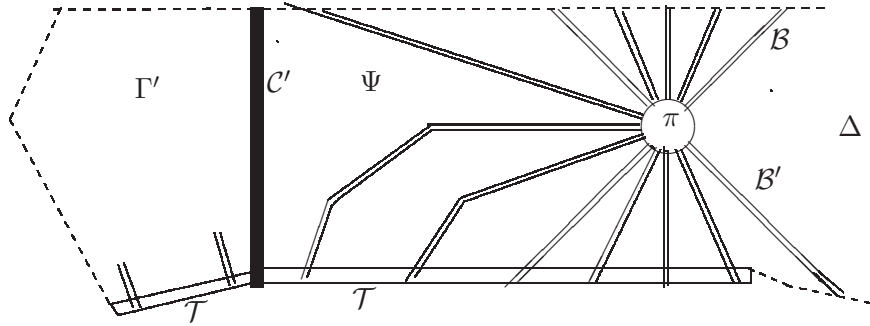
12.1 Solid diagrams

Let π be a hub in a diagram Δ , connected with the boundary $\partial\Delta$ by t -spokes \mathcal{B} and \mathcal{B}' . We denote by $cl(\pi, \mathcal{B}, \mathcal{B}')$ the subdiagram bounded by these spokes (and including them) and by subpaths of the boundaries of Δ and π , and call this subdiagram a *clove* if it has no hubs.

Lemma 12.1. (a) *Let $\Psi = cl(\pi, \mathcal{B}, \mathcal{B}')$ be a clove in a reduced diagram Δ . Assume that Ψ contains a rim θ -band \mathcal{T} , which crosses neither \mathcal{B} nor \mathcal{B}' , and every rim θ -band of Ψ with this property has at least $2LN$ q -cells. Then there is a maximal q -band \mathcal{C} in Ψ and a subcomb Γ with handle \mathcal{C} such that the base width of Γ is $15N$ and no q -band of Γ is a subband of a spoke of Δ .*

(b) *Assume that a reduced diagram Δ contains cells but has no hubs. Then either it has a rim band of base width $< 2LN$ or it has a subcomb of base width $15N$.*

Proof. (a) Since (1) a hub has LN spokes, (2) no q -band of Ψ intersects \mathcal{T} twice by Lemma 5.6, (3) \mathcal{T} has at least $2LN$ q -cells, and (4) $L > 30$, there exists a maximal q -band \mathcal{C}' such that a subdiagram Γ' separated from Ψ by \mathcal{C}' contains no edges of the spokes of π and the part of \mathcal{T} belonging to Γ' has at least $15N$ q -cells.



If Γ' is not a comb, and so a maximal θ -band of it does not cross \mathcal{C}' , then Γ' must contain another rim band \mathcal{T}' having at least $2LN$ q -cells by the assumption of the lemma. This makes possible to find a subdiagram Γ'' of Γ' such that a part of \mathcal{T}' is a rim band of Γ'' containing at least $LN > 15N$ q -cells, and Γ'' does not contain \mathcal{C}' . Since $\text{Area}(\Gamma') > \text{Area}(\Gamma'') > \dots$, such a procedure must stop. Hence, for some i , we obtain a subcomb $\Gamma^{(i)}$ of width $b \geq 15N$ intersected by no spokes. If $b > 15N$, then a derived subcomb of it has width $b - 1 \geq 15N$. Finally we obtain the desired Γ .

(b) The proof is easier than that for (a): one should just ignore the hub. \square

We call a minimal diagram *solid* if it has no rim θ -bands of base width $\leq 2LN$, no subcombs of base width $15N$ and no one-Step subcombs whose handles are $t^{\pm 1}$ - or $(t')^{\pm 1}$ -bands. Here we focus on solid diagrams since the proof of Theorem 1.1 will be reduced to them in the next section.

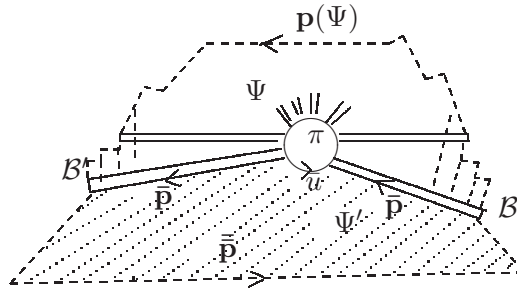
For a clove $\Psi = cl(\pi, \mathcal{B}, \mathcal{B}')$ in a diagram Δ , we denote by $\mathbf{p}(\Psi)$ the common subpath of $\partial\Psi$ and $\partial\Delta$ starting with the q -edge of \mathcal{B} and ending with the q -edge of \mathcal{B}' .

Lemma 12.2. *Let $\Psi = cl(\pi, \mathcal{B}, \mathcal{B}')$ be a clove in a solid diagram Δ . Then every maximal θ -band of Ψ crosses either \mathcal{B} or \mathcal{B}' ; the base width of any θ -band of Ψ is less than $2LN$, and $\text{Area}(\Psi) \leq (2LN(h + h') + \delta^{-1}|\mathbf{p}(\Psi)|)(h + h')$, where h and h' are the lengths of the bands \mathcal{B} and \mathcal{B}' , respectively.*

Proof. If the first claim were wrong, then one could find a rim θ -band which crosses neither \mathcal{B} nor \mathcal{B}' . Then by Lemma 12.1 (a), either first or the second condition in the definition of solid diagram would be violated, a contradiction. Thus the first statement of the lemma is proved, and Ψ has at most $h + h'$ maximal θ -bands.

Now we consider a maximal θ -band \mathcal{T} in Ψ (\mathcal{T} is not an annulus by Lemma 5.6). If its base width is at least $2LN$, then there is a maximal q -band \mathcal{C} intersecting \mathcal{T} which does not start/end on the hub π because the number of spokes starting on the same hub cell is LN . Moreover, as in the proof of Lemma 12.1, one can select \mathcal{C} so that \mathcal{C} separates a comb of base width $> (2LN - LN)/2 \geq 15N$ from Ψ , contrary to the assumption that Δ is solid. Thus, the base width of \mathcal{T} is less than $2LN$. Therefore the number of (θ, q) -cells in Ψ is at most $2LN \times (h + h')$. Every (θ, q) -cell has at most two a -edges by (iii). Hence the number of maximal a -bands starting and ending on the (θ, q) -cells of Ψ (but not on $\partial\Delta$) is at most $2LN(h + h')$. Their lengths do not exceed $h + h'$ by Lemma 5.6 since the number of maximal θ -bands in Ψ is at most $h + h'$. Thus the total area of these a -bands does not exceed $2LN(h + h')^2$. Arbitrary other maximal a -band and a maximal q -band of Ψ is also of length at most $h + h'$ by the same reason, but it has at least one edge on $\mathbf{p}(\Psi)$. Therefore their total area is at most $(h + h')(|\mathbf{p}(\Psi)|_a + |\mathbf{p}(\Psi)|_q) \leq \delta^{-1}|\mathbf{p}(\Psi)|$ by Lemma 5.20 (d). Since every cell of Ψ belongs to a θ -band, every (θ, a) -cell belongs to a -band and every maximal a -band starts and ends on a (θ, q) -cell or on $\partial\Delta$, the sum of these two inequalities gives the inequality from the lemma. \square

Let Ψ be a clove at a hub π in a solid diagram Δ . Assume that it has more than L t - and t' -spokes. (Recall that $\partial\pi$ has $2L$ t - and t' -edges.) Then we denote by $\bar{\Delta}$ the subdiagram formed by π and Ψ , and denote by $\bar{\mathbf{p}}$ the path $\mathbf{top}(\mathcal{B})\mathbf{u}^{-1}\mathbf{bot}(\mathcal{B}')^{-1}$, where \mathbf{u} is a subpath on $\partial\pi$, such that $\bar{\mathbf{p}}$ separate $\bar{\Delta}$ from the remaining subdiagram Ψ' of Δ . It follows that the total number of t - and t' -edges in \mathbf{u} is less than L , $|\mathbf{u}| < LN$, and the number of t - and t' -edges in $\mathbf{p}(\Psi)$ is at least $L + 1$.



Lemma 12.3. *If $|\partial\Delta| = n$ and $|\mathbf{p}(\Psi)| \geq 2LN \max(h, h')$, then, in the preceding notation, $|\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}| > 0$ and*

$$\text{Area}(\bar{\Delta}) \leq c_4(\mu(\Delta) - \mu(\Psi')) + c_5(\nu_J(\Delta) - \nu_J(\Psi')) + c_6n(n - |\partial\Psi'|)$$

Proof. Let us present $\partial\Psi'$ in the form $\bar{\mathbf{p}}\bar{\mathbf{p}}$. By Lemma 5.20(b), $|\partial\Psi'| \leq |\bar{\mathbf{p}}| + |\bar{\mathbf{p}}|$ and $n = |\mathbf{p}(\Psi)| + |\bar{\mathbf{p}}|$ since the first and the last edges of $\mathbf{p}(\Psi)$ are q -edges. Hence

$$n - |\partial\Psi'| \geq |\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}| \quad (12.64)$$

Note that by the definition of Ψ , we have $|\bar{\mathbf{p}}| \leq h + h' + |u| \leq h + h' + LN - 1$, and $|\mathbf{p}(\Psi)| \geq LN + 1$. Therefore in case $h = h' = 0$, we have $|\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}| \geq \max(2, \delta|\mathbf{p}(\Psi)|)$ by Lemma 5.20(d), since $\delta^{-1} \geq LN$. If $\max(h, h') \geq 1$, then by the second assumption of the lemma $\frac{3}{4}|\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}| > 1.5LN \max(h, h') - 2\max(h, h') - LN \geq 2$, and since $\delta < 1/4$, in any case we obtain

$$|\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}| \geq \max(\delta|\mathbf{p}(\Psi)|, 2) \quad (12.65)$$

Inequalities (12.64), (12.65), and the assumption of the lemma on $|\mathbf{p}(\Psi)|$ imply

$$|\mathbf{p}(\Psi)| \leq \delta^{-1}(n - |\partial\Psi'|), \quad n - |\partial\Psi'| \geq 2, \quad \text{and} \quad h + h' \leq n/LN \quad (12.66)$$

Now by Lemma 12.2 and Inequalities (12.66), we have

$$\begin{aligned} \text{Area}(\bar{\Delta}) &= \text{Area}(\Psi) + 1 \leq (2LN(h + h') + \delta^{-1}|\mathbf{p}(\Psi)|)(h + h') + 1 \leq \\ &(2|\mathbf{p}(\Psi)| + \delta^{-1}|\mathbf{p}(\Psi)|)(h + h') + 1 \leq (2 + \delta^{-1})\delta^{-1}(n - |\partial\Psi'|)(h + h') + 1 \\ &\leq 3(\delta^2LN)^{-1}(n - |\partial\Psi'|)n \leq c_6n(n - |\partial\Psi'|)/2 \end{aligned} \quad (12.67)$$

Recall now that in the definition of κ - and λ -mixtures, the middle point of every boundary q -edge is a black bead of the necklace on the boundary of diagram, and every white bead is a middle point of a boundary θ -edge (see Section 6 for details). It follows that $\kappa(\Delta) - \kappa(\Psi') \geq -|\bar{\mathbf{p}}|n$ and $\lambda(\Delta) - \lambda(\Psi') \geq -|\bar{\mathbf{p}}|n$ because new pairs of white beads (o, o') separated by black beads can appear in the necklace on $\partial\Psi'$ (in comparison with the necklace on $\partial\Delta$) only if one of the beads o, o' belongs to $\bar{\mathbf{p}}$. Hence by (12.65),

$$\min(\kappa(\Delta) - \kappa(\Psi'), \lambda(\Delta) - \lambda(\Psi')) \geq -\delta^{-1}(|\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}|)n,$$

and therefore by (12.64),

$$\begin{aligned} c_4(\mu(\Delta) - \mu(\Psi')) &= c_4((c_0\kappa(\Delta) + \lambda(\Delta)) - (c_0\kappa(\Psi') + \lambda(\Psi'))) = c_4(c_0(\kappa(\Delta) - \\ &\kappa(\Psi')) + (\lambda(\Delta) - \lambda(\Psi'))) \geq -c_4(c_0 + 1)\delta^{-1}(|\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}|)n \geq -c_6n(n - |\partial\Psi'|)/4 \end{aligned} \quad (12.68)$$

Recall also that the number of t - and t' -edges in the path $\bar{\mathbf{p}}$ (or in the path \mathbf{u}) does not exceed the similar number for $\mathbf{p}(\Psi)$. Therefore any two white beads o, o' of the ν -necklace on $\partial\Delta$, provided they both belong to $\mathbf{p}(\Psi)$, are separated by at least the same number of black beads in the ν -necklace for Δ as in the ν -necklace for Ψ' (either the clockwise arc $o - o'$ includes $\mathbf{p}(\Psi)$ or not). So such a pair contributes to $\nu_J(\Delta)$ at least the amount it contributes to $\nu_J(\Psi')$. Thus, to estimate $\nu_J(\Delta) - \nu_J(\Psi')$ from below, it suffices to consider the contribution to $\nu_J(\Psi')$ for the pairs o, o' , where one of the two beads lies on $\bar{\mathbf{p}}$. Then the argument we used above for κ - and λ -mixtures, yields $\nu_J(\Delta) - \nu_J(\Psi') \geq -J\delta^{-1}(|\mathbf{p}(\Psi)| - |\bar{\mathbf{p}}|)n$. Hence $c_5(\nu_J(\Delta) - \nu_J(\Psi')) \geq -c_6n(n - |\partial\Psi'|)/4$ by (12.64) since $c_6 > 4J\delta^{-1}c_5$. This inequality together with (12.67) and (12.68) prove the lemma. \square

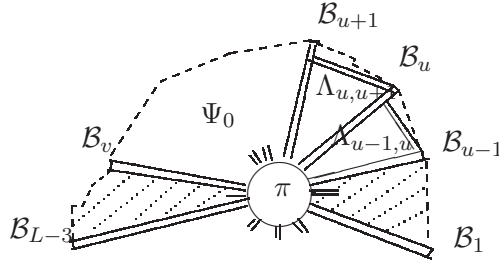
A clove $cl(\pi, \mathcal{B}, \mathcal{B}')$ will be called a *crescent* if

- (1) it contains $l \geq L - 20 > L/2$ consecutive $t^{\pm 1}$ -spokes $\mathcal{C}_1 = \mathcal{B}, \mathcal{C}_2, \dots, \mathcal{C}_l = \mathcal{B}'$ connecting $\partial\Delta$ and $\partial\pi$;
- (2) every maximal θ -band of this clove crosses either \mathcal{C}_1 or \mathcal{C}_l ; moreover, either all maximal θ -bands of Ψ cross \mathcal{C}_1 , or all of them cross \mathcal{C}_l , or there exists $i, 2 \leq i \leq l - 2$ such that the θ -bands crossing \mathcal{C}_l but not \mathcal{C}_1 , do not cross \mathcal{C}_i , and the θ -bands crossing \mathcal{C}_1 but not \mathcal{C}_l , do not cross \mathcal{C}_{i+1} ;
- (3) every maximal (12)- or (23)-band of Ψ crossing \mathcal{C}_1 (crossing \mathcal{C}_l) also crosses \mathcal{C}_2 (crosses \mathcal{C}_{l-1}), and every spoke of the clove is crossed by at most 3 (12)- or (23)-bands.

Lemma 12.4. *Assume a solid diagram Δ has a hub. Then it contains a crescent $\Psi = cl(\pi, \mathcal{C}_1, \mathcal{C}_l)$ such that the cloves $cl(\pi, \mathcal{C}_2, \mathcal{C}_l)$ and $cl(\pi, \mathcal{C}_1, \mathcal{C}_{l-1})$ are also crescents.*

Proof. We consider a hub π provided by Lemma 5.18. There are consecutive maximal $t^{\pm 1}$ -bands $\mathcal{B}_1, \dots, \mathcal{B}_{L-3}$ connecting (counter-clockwise) $\partial\Delta$ and $\partial\pi$, such that the subdiagram $\bar{\Psi}$ bounded by $\mathcal{B}_1, \mathcal{B}_{L-3}, \partial\Delta$, and $\partial\pi$ contains all these $t^{\pm 1}$ -bands but does not contain hubs. Observe that by Lemma 12.1, every maximal θ -band of $\bar{\Psi} = cl(\pi, \mathcal{B}_1, \mathcal{B}_{L-3})$ crosses either \mathcal{B}_1 or \mathcal{B}_{L-3} because Δ is solid.

Consider a subdiagram $\Psi(0) = cl(\pi, \mathcal{B}_u, \mathcal{B}_v)$ of $\bar{\Psi}$ with $u - v = L - k$ for some $k, 6 \leq k < L, u > 1, v < L - 3$. Every maximal θ -band of $\Psi(0)$ crosses either \mathcal{B}_u or \mathcal{B}_v .



Let $\Lambda_{u-1,u}$ (let $\Lambda_{u+1,u}$) be the trapezium formed by all θ -bands starting on \mathcal{B}_{u-1} (starting on \mathcal{B}_{u+1} , resp.) and ending on \mathcal{B}_u . It contains M_4 -accepting subtrapezia with the same histories, and so there are at most 3 (12)- and (23)-bands among them by (x) and (viii); similarly for $\Lambda_{v,v+1}$ and $\Lambda_{v,v-1}$. Since \mathcal{B}_u (resp. \mathcal{B}_v) must belong to one of trapezia $\Lambda_{u-1,u}$ and $\Lambda_{u+1,u}$ (to one of $\Lambda_{v,v+1}$ and $\Lambda_{v,v-1}$, resp.) the number of maximal ((12), t)- and ((23), t)-cells in \mathcal{B}_u (in \mathcal{B}_v), and therefore in any spoke of $\Psi(0)$, is at most 3, and the number of maximal (12)- and (23)-bands in $\Psi(0)$ is at most 6. We want to obtain a clove $\Psi(1) = cl(\pi, \mathcal{B}_{u'}, \mathcal{B}_{v'})$ ($u' \geq u, v' \leq v$) applying one of the following transitions changing the pair (u, v) .

(a) If a (12)- or (23)-band crosses \mathcal{B}_u but not \mathcal{B}_{u+2} (\mathcal{B}_v but not \mathcal{B}_{v-2}), then we set $u' = u + 2, v' = v$ ($u' = u, v' = v - 2$).

(b) Notice that either all maximal θ -bands of $\Psi(0)$ cross \mathcal{B}_u or all of them cross \mathcal{B}_v , or there exists i ($u \leq i < v$) such that the θ -bands crossing \mathcal{B}_v but not \mathcal{B}_u , do not cross \mathcal{B}_i , and the θ -bands crossing \mathcal{B}_u but not \mathcal{B}_v , do not cross \mathcal{B}_{i+1} . If in the latter case $i \geq v - 2$, then we set $u' = u, v' = v - 2$. Similarly we set $u' = u + 2, v' = v$ if $i \leq u + 1$.

After a transition (a) or (b), we obtain a clove $\Psi(1) = cl(\pi, \mathcal{B}_{u'}, \mathcal{B}_{v'})$ with $v' - u' \geq L - k'$, where $k' = k + 2$. Let us start with $u = 2$ and $v = L - 4$ (i.e., $k = 6$) and apply a maximal series of transitions of type (a). Since every transition of type (a) removes a maximal (12)- or (23)-band and the number of such bands in $\Psi(0)$ is at most 6, the

length of the series is also at most 6. Then, if possible, we apply a transition of type (b). Note that no transition of types (a) and (b) is applicable to a clove $\Psi(m)$ with $m \leq 7$. We have $k^{(m)} \leq 6 + 2 \times 7 = 20$.

It remains to set $\mathcal{C}_1 = \mathcal{B}_{u^{(m)}}$, and $\mathcal{C}_l = \mathcal{B}_{v^{(m)}}$. Then $\Psi = \Psi(m) = cl(\pi, \mathcal{C}_1, \mathcal{C}_l)$ satisfies the conditions (1)-(3) from the definition of crescent. Indeed condition (1) holds since $k^{(m)} - 1 \leq 19$ and $L > 40$, condition (2) (condition (3)) holds since no transition of type (b) (of type (a), resp.) is applicable to Ψ . The cloves $cl(\pi, \mathcal{C}_2, \mathcal{C}_l)$ and $cl(\pi, \mathcal{C}_1, \mathcal{C}_{l-1})$ are also crescents since (1) $20 \leq 41/2$ and (2),(3) no transitions of type (b) and (a) are applicable to Ψ . \square

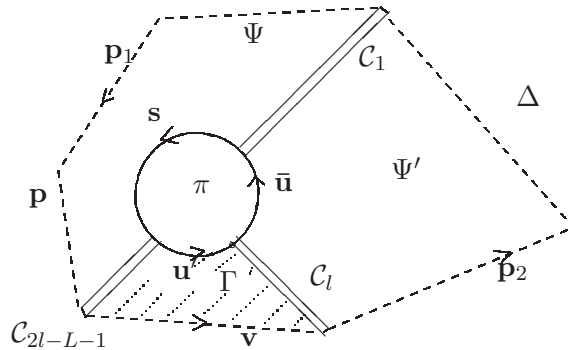
Lemma 12.5. *The number of maximal t - and t' -bands in a crescent Ψ is less than $13LN < J/2$.*

Proof. The number of maximal t - and t' -bands ending on π is less than $2L$. Any other maximal t - or t' -band \mathcal{C} starts and ends on the subpath $\mathbf{p}(\Psi)$ of the boundary of Δ and separates a subdiagram Φ from the crescent $\Psi = cl(\pi, \mathcal{B}, \mathcal{B}')$ such that Φ contains \mathcal{C} but has no cells from \mathcal{B} or \mathcal{B}' . However by the definition of crescent, every cell from Φ is connected with either \mathcal{B} or \mathcal{B}' by a θ -band. It follows that every maximal θ -band of Φ has to cross \mathcal{C} , i.e., Φ is a subcomb of Δ with handle \mathcal{C} . This subcomb is not one-Step since Δ is a solid diagram, and therefore \mathcal{C} has either (12)- or (23)-cell. In other words, it crosses one of the maximal θ -bands crossing \mathcal{C}_1 or \mathcal{C}_l and corresponding to one of the rules (12), (13). The number of such θ -bands is at most 6 by Properties (3) and (2) from the definition of crescent. Their base widths $< 2LN$ by Lemma 12.2, and so the number of maximal t - and t' -bands \mathcal{C} which have no ends on π , is less than $12LN$. The lemma is proved because $12LN + 2L < 13LN$. \square

12.2 Surgery removing a hub

When we induct on the number of hubs, we want to cut up a subdiagram Δ_1 with one hub so that $\text{Area}(\Delta_1)$ to be bounded in ‘quadratic terms’ (as we did earlier for subcombs). The estimates of Lemmas 12.8 and 12.9 will be applied in the final Section 13.

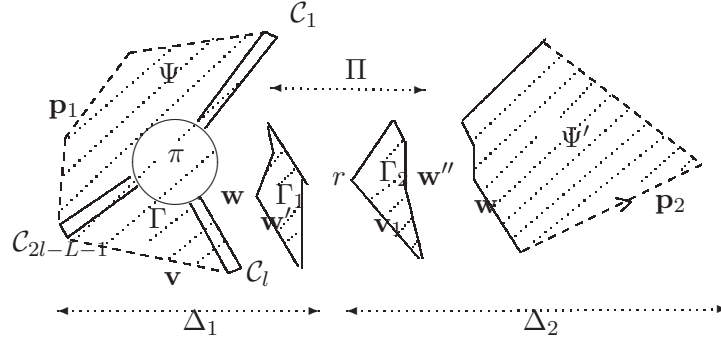
In this subsection we consider a solid diagram Δ with the hub π and the crescent $\Psi = cl(\pi, \mathcal{C}_1, \mathcal{C}_l)$ provided by Lemma 12.4. We will assume that the t -spokes $\mathcal{C}_1, \mathcal{C}_l$ are enumerated counter-clockwise with respect to the hub π , and the histories H_1, \dots, H_l of $\mathcal{C}_1, \mathcal{C}_l$ are read towards π . We have $\partial\Psi = \mathbf{top}(\mathcal{C}_1)^{-1}\mathbf{pbot}(\mathcal{C}_l)\mathbf{s}^{-1}$, where $\mathbf{p} = \mathbf{p}(\Psi)$ and \mathbf{s} is a subpath in $\partial\pi$. Then $\partial\Delta = \mathbf{pp}_2$. The diagram Δ is the union of Ψ, π , and the remaining subdiagram Ψ' . Let now $h_i = \|H_i\|$, $i = 1, \dots, l$.



Without loss of generality, we will assume further that $h_1 \geq h_l$ for Ψ . Under this assumption, we will use the following special surgery for Δ . Denote by e_j the common t -edge of $\partial\mathcal{C}_j$ and $\partial\pi$. Consider the reduced subpath $e_{2l-L-1}\mathbf{u}'e_l$ of \mathbf{s} . Denote by Γ the subdiagram without hubs bounded by $\mathbf{bot}(\mathcal{C}_{2l-L-1})^{-1}(\mathbf{v})\mathbf{top}(\mathcal{C}_l)(\mathbf{u}')^{-1}$, where \mathbf{v} is a subpath of $\partial\Delta$. We have $\mathbf{p}(\Psi) = \mathbf{p} = \mathbf{p}_1\mathbf{v}f$ for some \mathbf{p}_1 , where f is the common edge of $\partial\mathcal{C}_l$ end $\partial\Delta$.

There is a reduced path $e_1^{-1}\bar{\mathbf{u}}e_l^{-1}$, where $(\bar{\mathbf{u}})^{-1}$ is a subpath of $\partial\pi$. Then the paths $\mathbf{w}_1 = \mathbf{top}(\mathcal{C}_l)(\mathbf{u}')^{-1}$ is obtained from $\mathbf{w} = \mathbf{bot}(\mathcal{C}_l)\bar{\mathbf{u}}^{-1}$ by a t_i -reflection since \mathcal{C}_l is a t_i -band for some i (see definitions in Remark 5.3). Therefore the following surgery is possible.

- (1) Cut Δ along \mathbf{w} .
- (2) Construct a diagram Γ_1 obtained from Γ by the t_i -reflection (see Remark 5.3) and take a standard mirror copy Γ_2 of Γ_1 (where the mirror edges have equal labels). Glue Γ_1 and Γ_2 together along the path \mathbf{r} obtained by the t_i -reflection from $\mathbf{bot}(\mathcal{C}_{2l-L-1})^{-1}\mathbf{v}$, and obtain a diagram Π with boundary $\mathbf{w}'\mathbf{w}''$, where $\text{Lab}(\mathbf{w}') \equiv \text{Lab}((\mathbf{w}'')^{-1}) \equiv \text{Lab}(\mathbf{w}^{-1})$.



- (3) Insert Π in the hole of Δ obtained after step (1).
- (4) Cut up the obtained disc diagram along $\mathbf{top}(\mathcal{C}_1)\mathbf{r}$, and obtain two diagrams Δ_1 and Δ_2 , where Δ_1 is a minimal diagram with the same boundary label as the union of Ψ , π and Γ_1 , and Δ_2 is a union of Ψ' and Γ_2 .
- (5) Let H_0 be the history of the maximal trapezium bounded by \mathcal{C}_1 and \mathcal{C}_{2l-L-1} in Ψ (it is the filling trapezium $Tp(\mathcal{C}_{2l-L-1}, \mathcal{C}_1)$ if every maximal θ -band crossing \mathcal{C}_{2l-L-1} also crosses \mathcal{C}_1), and so H_0 is a suffix of both H_1 and H_{2l-L-1} . Therefore $2h_0 = 2||H_0||$ letters can be canceled in the product $\text{Lab}(\mathbf{top}(\mathcal{C}_1))\text{Lab}(\mathbf{r})$. And so we shorten the corresponding part of the boundary of Δ_2 by $2h_0$ edges and replace the obtained diagram by a minimal diagram Δ' .

Thus the boundary of Δ' is $\mathbf{p}_3\mathbf{p}_2$, where $\mathbf{p}_3 = \mathbf{x}\mathbf{v}'$, \mathbf{v}' is obtained by the t_i -reflection of \mathbf{v} , and $|\mathbf{x}| = |\mathbf{x}|_\theta = h_1 + h_{2l-L-1} - 2h_0$. Since the path \mathbf{p}_1 has at least $(h_1 - h_0) + (h_{2l-L+1} - h_0)$ θ -edges (the ends of maximal θ -bands which cross \mathcal{C}_1 but do not cross \mathcal{C}_{2l-L-1} , and vice versa) and also has q -edges, we have

$$|\mathbf{x}| = |\mathbf{x}|_\theta \leq |\mathbf{p}_1|_\theta \leq |\mathbf{p}_1| - 1 \quad (12.69)$$

Moreover using the maximal θ -bands crossing \mathcal{C}_1 and \mathcal{C}_l in the crescent Ψ , one can for every θ -edges of \mathbf{x} , find a θ -edge of \mathbf{p}_1 corresponding to the same rule $\theta^{\pm 1}$, and the obtaining mapping from the set of (non-oriented) edges of \mathbf{x} to the set of edges of \mathbf{p}_1 is injective.

Lemma 12.6. *With the preceding notation, we have (a) $|\partial\Delta'| \leq |\partial\Delta| - 1$; (b) $\kappa(\Delta') \leq \kappa(\Delta)$; (c) $\lambda(\Delta') \leq \lambda(\Delta) + 2h_1^2$; (d) $\nu_J(\Delta') \leq \nu_J(\Delta)$.*

Proof. (a) Since $|\mathbf{v}'| = |\mathbf{v}|$, the statement (a) follows from Inequality (12.69) and Lemma 5.20 (b).

(b) Recall that \mathbf{v}' is constructed as the t_i -reflection of \mathbf{v} . Thus, when passing from the boundary label of Δ to the boundary label of Δ' , we, in essence, just replace $\text{Lab}(\mathbf{p}_1)$ by $\text{Lab}(\mathbf{x})$. But \mathbf{x} has no q -edges, and so it has no black beads (see the definition of the κ -mixture of a diagram), and the number of white beads of \mathbf{x} is at most the number of white beads on \mathbf{p}_1 by (12.69). Therefore $\kappa(\Delta') \leq \kappa(\Delta)$ by Lemma 6.1 (Parts b,c).

(d) Similarly, using the fact that the path \mathbf{x} has no t - or t' -edges one concludes that $\nu_J(\Delta') \leq \nu_J(\Delta)$.

(c) The remark made before the formulation of the lemma, allows us to obtain an injective mapping from the set of white beads of the λ -necklace O' for $\partial\Delta'$ to the set of white beads of the λ -necklace O on $\partial\Delta$, so that the beads from \mathbf{x} map to the beads on \mathbf{p}_1 . It follows that if o, o' are two white beads from O' , but not both on \mathbf{x} , and they are separated by a black bead in O' , then the corresponding white beads of O are also separated by a black bead. (We take into account that \mathbf{p}_1 starts and ends with q -edges having black beads by the definition of the necklace O .) Therefore to estimate the difference $\lambda(\Delta) - \lambda(\Delta')$ from below, we may consider only the pairs of white beads of O' , where both o and o' belong to \mathbf{x} , whence $\lambda(\Delta) - \lambda(\Delta') \geq -\lambda(\mathbf{x})$. By Lemma 6.2 (a), $\lambda(\mathbf{x}) < |\mathbf{x}|^2/2 \leq (2h_1)^2/2$ and claim (c) is proved. \square

Remark 12.7. (1) The surgery described before the formulation of Lemma 12.6, can be also done for the original clove $cl(\pi, \mathcal{B}_1, \mathcal{B}_{L-3})$ even if one does not assume that Δ is a solid diagram. In this case again, exactly as in the proof of Lemma 12.6, we obtain the inequality $|\partial\Delta'| \leq |\partial\Delta| - 1$.

(2) Assume that Γ is a subcomb of a diagram Δ , and the handle of Γ is a $t^{\pm 1}$ or $(t')^{\pm 1}$ -band, $\mathbf{y} = \mathbf{y}^\Gamma$, and $\Delta' = \Delta \setminus \Gamma$. Then by Lemma 5.6, we have a preserving the order bijective mapping from the set of θ -edges of \mathbf{y}^{-1} to the set of the θ -edges of $\mathbf{z} = \mathbf{z}^\Gamma$. Then arguing exactly as in the proof of Part (c) of Lemma 12.6, we get $\lambda(\Delta) - \lambda(\Delta') \geq -\lambda(\mathbf{y}) > -|\mathbf{y}|^2/2$. Hence

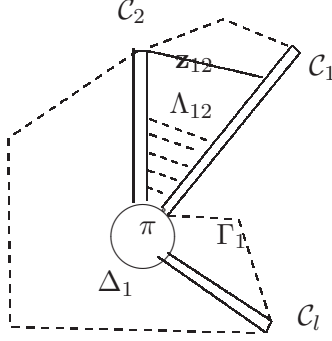
$$\mu(\Delta) - \mu(\Delta') > -|\mathbf{y}|^2/2,$$

by the definition of $\mu(\cdot)$ and Lemmas 7.2 (a) and 7.3 (a).

Lemma 12.8. *Assume that $n = |\partial\Delta|$, h_1 does not belong to any interval $(T_i, 9T_i)$ ($i = 1, 2, \dots$) and $h_2 > (1 - \frac{1}{30N})h_1$. Then, with the notation of Lemma 12.6, we have*

$$\text{Area}(\Delta_1) \leq c_4(\mu(\Delta) - \mu(\Delta')) + c_5(\nu_J(\Delta) - \nu_J(\Delta')) + c_6n(n - |\partial\Delta'|)$$

Proof. Since $h_1 = \max_{i=1}^l h_i$, the condition (2) from the definition of crescent implies that every maximal θ -band crossing the $t^{\pm 1}$ -band \mathcal{C}_2 in the crescent Ψ , has to cross \mathcal{C}_1 as well. Therefore we can consider the trapezium $\Lambda_{12} = Tp(\mathcal{C}_2, \mathcal{C}_1)$ of height h_2 between \mathcal{C}_1 and \mathcal{C}_2 . The bottom path \mathbf{z}_{12} of Λ_{12} , must be of a -length at least $h_2/6$ by (xix), since h_2 does not belong to any interval $(T_i, 9T_i)$.



Recall that the diagram Δ is solid, and therefore the clove $cl(\pi, \mathcal{C}_1, \mathcal{C}_2)$ has $h_1 - h_2 < \frac{h_1}{30N}$ maximal θ -bands outside Λ_{12} . Hence the maximal a -bands starting on \mathbf{z}_{12} can end outside of $\Lambda_{1,2}$ on at most N (θ, q) -cells of each of the $< \frac{h_1}{30N}$ θ -bands. Hence by (iii) (a), at least $|\mathbf{z}_{12}|_a - h_1/15$ a -bands starting on \mathbf{z}_{12} end on \mathbf{p}_1 , and so $|\mathbf{p}_1|_a > h_1(1/6 - 1/15) = h_1/10$.

Therefore by Lemma 5.20 (a) and Inequality (12.69), we obtain

$$|\mathbf{p}| = |\mathbf{p}_1| + |\mathbf{v}| > |\mathbf{p}_1|_\theta + \delta' h_1/10 + |\mathbf{v}| \geq |\mathbf{x}| + |\mathbf{v}| + \delta' h_1/10 \geq |\mathbf{p}_3| + \delta' h_1/10,$$

and so $|\mathbf{p}| - |\mathbf{p}_3| > \delta' h_1/10$. Thus

$$h_1 < 10(\delta')^{-1}(|\mathbf{p}| - |\mathbf{p}_3|) \quad (12.70)$$

We have by (12.70) and Lemma 12.2:

$$\begin{aligned} \text{Area}(\Delta_1) &\leq 2\text{Area}(\Psi) + 1 \leq 4(2LN(h_1 + h_l) + \delta^{-1}|\mathbf{p}|)h_1 + 1 \leq 16LNh_1^2 + 5\delta^{-1}|\mathbf{p}|h_1 \\ &< 16LN \times 100(\delta')^{-2}(|\mathbf{p}| - |\mathbf{p}_3|)^2 + 5\delta^{-1}|\mathbf{p}| \times 10(\delta')^{-1}(|\mathbf{p}| - |\mathbf{p}_3|) < c_6|\mathbf{p}|(|\mathbf{p}| - |\mathbf{p}_3|)/2 \end{aligned} \quad (12.71)$$

since $|\mathbf{p}_3| \leq |\mathbf{p}|$. By Lemma 12.6, $\lambda(\Delta) - \lambda(\Delta') \geq -2h_1^2$, and therefore by (12.70),

$$c_4(\lambda(\Delta) - \lambda(\Delta')) \geq -2c_4h_1^2 \geq -2c_4(10)^2(\delta')^{-2}(|\mathbf{p}| - |\mathbf{p}_3|)^2 \geq -c_6|\mathbf{p}|(|\mathbf{p}| - |\mathbf{p}_3|)/2 \quad (12.72)$$

since $c_6 > 400(\delta')^{-2}c_4$. Hence by (12.71) and (12.72),

$$\text{Area}(\Delta_1) \leq c_6|\mathbf{p}|(|\mathbf{p}| - |\mathbf{p}_3|) + c_4(\lambda(\Delta) - \lambda(\Delta')) \leq c_6n(n - |\partial\Delta'|) + c_4(\lambda(\Delta) - \lambda(\Delta'))$$

as $|\mathbf{p}| - |\mathbf{p}_3| \leq n - |\partial\Delta'|$. Now the statement follows from Lemma 12.6 (b,d). \square

Let now $\Psi_{2,l}$ be the part of the crescent Ψ between \mathcal{C}_2 and \mathcal{C}_l . By Lemma 12.4, $\Psi_{2,l}$ is a crescent too. For the crescent $\Psi_{2,l}$, one can define the analogs of $\mathbf{p}, \mathbf{p}_1, \mathbf{p}_3, \mathbf{v}, \mathbf{v}', \Delta_1$ and Δ' introduced earlier for the crescent Ψ . We denote them by $\mathbf{p}(0), \mathbf{p}_1(0), \mathbf{p}_3(0), \mathbf{v}(0), \mathbf{v}'(0), \Delta_1(0)$ and $\Delta'(0)$, respectively.

The substitution of Ψ by $\Psi_{2,l}$ in Lemma 12.2, gives us

$$\text{Area}(\Psi_{2,l}) \leq (h_2 + h_l)(2LN(h_2 + h_l) + \delta^{-1}|\mathbf{p}(0)|) \quad (12.73)$$

Lemma 12.9. Assume that $|\mathbf{p}(0)| \leq 2LN \max(h_2, h_l)$, $h_2 < (1 - \frac{1}{30N})h_1$, and $\max(h_2, h_l)$ does not belong to any interval $(T_i, 9T_i)$. Then the following inequality holds:

$$\text{Area}(\Delta_1(0)) \leq c_4(\mu(\Delta) - \mu(\Delta'(0))) + c_5(\nu_J(\Delta) - \nu_J(\Delta'(0))) + c_6|\partial\Delta|(|\partial\Delta| - |\partial\Delta'(0)|). \quad (12.74)$$

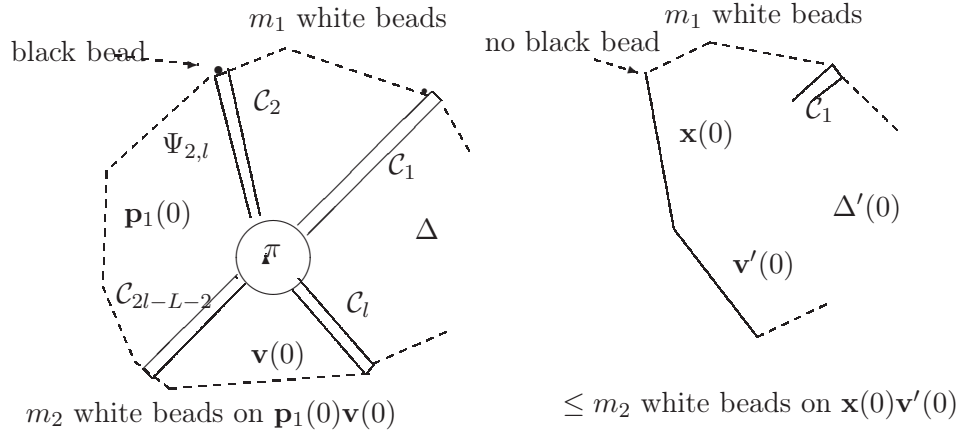
Proof. Assume that $h_0 \geq (1 - \frac{1}{30})h_{2,l}$, where $h_{2,l} = \max(h_2, h_l)$. Since $h_i \geq h_0$ for any $i = 1, \dots, l$, to complete the proof, it suffices to apply Lemma 12.8 to $\Delta_1(0)$. Then we assume that $h_0 < (1 - \frac{1}{30N})h_{2,l}$.

By Lemma 12.2 and the restriction on $|p(0)|$,

$$\begin{aligned} \text{Area}(\Delta_1(0)) &\leq 2\text{Area}(\Psi_{2,l}) + 1 \leq 4(2LN(h_2 + h_l) + \delta^{-1}|\mathbf{p}(0)|)h_{2,l} + 1 \\ &\leq 16LNh_{2,l}^2 + 4\delta^{-1}(2LN)^2h_{2,l}^2 \leq (\delta')^{-1}h_{2,l}^2 \end{aligned} \quad (12.75)$$

since $(\delta')^{-1} > 20\delta^{-1}L^2N^2$.

Now we want to estimate $\nu_J(\Delta) - \nu_J(\Delta'(0))$. For this aid, we observe that the common t -edge f_2 of the spoke \mathcal{C}_2 and $\partial\Delta$ separates at least $h_1 - h_2 = m_1$ θ -edges placed on \mathbf{p} between \mathcal{C}_1 and \mathcal{C}_2 and m_2 ones placed between \mathcal{C}_2 and \mathcal{C}_l , where $m_2 \geq \max(h_2 - h_0, h_l - h_0) \geq \frac{1}{30N}h_{2,l}$. Lemmas 12.5 and 6.1 (d) imply that one decreases $\nu_J(\Delta)$ at least by m_1m_2 when erasing the black bead on f_2 in the ν -necklace on $\partial\Delta$.



Nevertheless we do such erasing while passing from Δ to $\Delta'(0)$ since the path $\mathbf{x}(0)$ (replacing the path $\mathbf{p}_1(0)$ with edge f_2) has no t - or t' -edges and $\mathbf{v}'(0)$ is a copy of $\mathbf{v}(0)$. (We might erase some other black and white beads). Hence

$$\nu_J(\Delta) - \nu_J(\Delta'(0)) \geq m_1m_2 \geq \frac{1}{30N}h_1\left(\frac{1}{30N}\right)h_{2,l} \geq \frac{1}{(30N)^2}h_{2,l}^2 \quad (12.76)$$

The Inequalities (12.76) and (12.75) imply

$$\text{Area}(\Delta_1(0)) \leq c_5(\nu_J(\Delta) - \nu_J(\Delta'(0)))/2 \quad (12.77)$$

since $c_5 > 2000N^2(\delta')^{-1}$.

By Lemma 12.6 (b,c) for the diagrams Δ and $\Delta'(0)$, we have $\mu(\Delta) - \mu(\Delta'(0)) \geq \lambda(\Delta) - \lambda(\Delta'(0)) \geq -2h_{2,l}^2$, whence by (12.76), $0 \leq c_5(\nu_J(\Delta) - \nu_J(\Delta'(0)))/2 + c_4(\mu(\Delta) - \mu(\Delta'(0)))$ since $c_5 > 2000N^2c_4$. Adding this inequality with (12.77), we have

$$\text{Area}(\Delta_1(0)) \leq c_5(\nu_J(\Delta) - \nu_J(\Delta'(0))) + c_4(\mu(\Delta) - \mu(\Delta'(0)))$$

This implies Inequality (12.74) since the third summand at the right-hand side of (12.74) is positive by Lemma 12.6 (a) applied to the diagrams Δ and $\Delta'(0)$.

The notation of this subsection will also be used in the next section.

13 Almost quadratic upper bound

We denote by $g(n)$ the minimal function such that the height of any M_4 -accepting trapezium is at most $g(n)$ if the a -length of its bottom label does not exceed n . (Such upper bounds exist for every n by Properties (xvii) and (xviii).) Since $g(n)$ is non-decreasing, the auxiliary function $f(n) = n(8g(\delta^{-1}n^2)^2 + \delta^{-1}n^3)$ used in this section is also non-decreasing. For the beginning of this section, we need a crude upper bound for areas of diagrams.

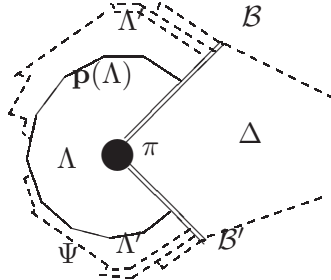
Lemma 13.1. *Let Δ be a minimal diagram of perimeter $\leq n$. Then $\text{Area}(\Delta) \leq f(n)$.*

Proof. Step 1. Assume that Δ has no hubs. Then we can use Lemma 5.6, and the total number of maximal q -band and maximal θ -bands of Δ is at most $n/2$. Hence the number of (q, θ) -cells is at most $n^2/16$. Since every maximal a -band ends either on the boundary $\partial\Delta$ or on a (q, θ) -cell, the number such bands is at most $(2 \times n^2/16 + \delta^{-1}n)/2$ by Lemma 5.20 (d) and (iii) (a). Each of these a -bands crosses at most $n/2$ θ -bands, and so their total area is at most $n^3/16 + \delta^{-1}n^2/4$. Therefore $\text{Area}(\Delta) \leq n^3/16 + \delta^{-1}n^2/4 + n^2/16 \leq \delta^{-1}n^3/3$.

Step 2. In any case, the number of hubs n_{hub} in Δ is at most $2n/LN$ by Lemma 5.19. To complete the proof of the lemma, it suffices to assume that $n_{hub} \geq 1$ and to prove by induction on n_{hub} that

$$\text{Area}(\Delta) \leq n_{hub}(4LN g(\delta^{-1}n^2)^2 + 6\delta^{-1}n^3) + \delta^{-1}n^3/3$$

There are a hub π and a clove $\Psi = cl(\pi, \mathcal{B}, \mathcal{B}')$ given by Lemma 5.18. Let Λ be the subdiagram of Ψ formed by the θ -bands of Ψ crossing both t -bands $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{B}' = \mathcal{B}_{L-3}$. Let the remaining part $\Lambda' = \Psi \setminus \Lambda$ be separated from Λ by a path $\mathbf{p}(\Lambda)$.



It follows from the choice of Λ that every maximal θ -band \mathcal{T} of Λ' starts or ends on $\partial\Delta$. Hence the number of such θ -bands is at most n . In the diagram Λ' , a θ -band

of Δ and a q -band have at most one common (q, θ) -cells by Lemma 5.6. Since the number of maximal q -bands of Λ' is at most $|\mathbf{p}(\Psi)| \leq n$, the number of (q, θ) -cells in Λ' is not greater than n^2 . Since every maximal a -band of Λ' starting on the path $\mathbf{p}(\Lambda)$ must end on one of these (q, θ) -cells or on $\partial\Delta$, the number of a -edges in $\mathbf{p}(\Lambda)$ is at most $2n^2 + \delta^{-1}n \leq 2\delta^{-1}n^2$ by Lemma 5.20 (d). Since by Lemma 5.6, a maximal a -band intersects a θ -band of the diagram Λ' at most once, there are at most $2\delta^{-1}n^3$ (θ, a) -cells in Λ' . Thus, $\text{Area}(\Lambda') \leq 2\delta^{-1}n^3 + n^2 \leq 2.5\delta^{-1}n^3$.

Since $|\mathbf{p}(\Lambda)|_a \leq 2\delta^{-1}n^2$ and Λ has $2(L-4)$ M_4 -accepting trapezia whose top labels are just copies of one of them (see Remark 5.3 and Lemma 5.10), the number of maximal θ -bands in Λ is at most $g((2\delta^{-1}n^2)(2L-8)^{-1}) \leq g(\delta^{-1}n^2)$. By (x) applied to $2L-8$ M_4 -accepting trapezia, the number of cells in any maximal θ -band of Λ does not exceed $LN + (2L-8)4g(\delta^{-1}n^2)$. Multiplying this number by the height of Λ , we obtain

$$\text{Area}(\Lambda) \leq ((8L-32)g(\delta^{-1}n^2) + LN)g(\delta^{-1}n^2) \leq 2LNg(\delta^{-1}n^2)^2.$$

Therefore $\text{Area}(\Psi) = \text{Area}(\Lambda) + \text{Area}(\Lambda') \leq 2LNg(\delta^{-1}n^2)^2 + 3\delta^{-1}n^3 - 1$.

Now we use the surgery from Remark 12.7 (1) and have $\text{Area}(\Delta_1) \leq 2\text{Area}(\Psi) + 1 \leq 4LNg(\delta^{-1}n^2)^2 + 6\delta^{-1}n^3$, and $|\partial\Delta'| \leq |\partial\Delta| - 1$. Since the number of hubs of Δ' is strictly less than this number for Δ , we have by the inductive hypothesis, $\text{Area}(\Delta') \leq (n_{\text{hub}} - 1)(4LNg(\delta^{-1}n^2)^2 + 6\delta^{-1}n^3) + (\delta')^{-1}n^3/3$, and therefore, by Lemma 5.2, as required,

$$\text{Area}(\Delta) \leq \text{Area}(\Delta_1) + \text{Area}(\Delta') \leq n_{\text{hub}}(4LNg(\delta^{-1}n^2)^2 + 6\delta^{-1}n^3) + (\delta')^{-1}n^3/3$$

□

The following lemma summarizes our efforts and ensures the main result.

Lemma 13.2. *Let the perimeter $n = |\partial\Delta|$ of a minimal diagram Δ satisfy inequality $f(n) \leq T_i$ for some i , where f is the function from Lemma 13.1. Then $\text{Area}(\Delta) \leq F(\Delta)$ for $F(\Delta) = F(\partial\Delta) = c_4\mu(\Delta) + c_5\nu_J(\Delta) + c_6n^2 + c_7n_Qf(T_{i-1})$, where n_Q is the number of q -edges in $\partial\Delta$.*

Proof. (1) If $\partial\Delta$ has no q -edges, then Δ has no q -edges by Lemmas 5.18 and 5.6. Then $\text{Area}(\Delta) < \delta^{-1}n^2 \leq c_6n^2$ because (1) a maximal θ -band and a maximal a -band have at least one common (θ, a) -cell, (2) Δ has no θ - and a -annuli and so every maximal θ - and a -band starts and ends on $\partial\Delta$ by Lemma 5.6, and (3) $|\partial\Delta|_a \leq |\partial\Delta| \leq \delta^{-1}n$ by Lemma 5.20 (a,d).

Thus we suppose $n_Q \geq 1$. By Lemma 13.1, $\text{Area}(\Delta) \leq f(T_{i-1})$ if $n \leq T_{i-1}$. Then we **may suppose $n > T_{i-1}$, $n \geq 1$ and prove the lemma by contradiction assuming further that Δ is a counter-example with minimal perimeter n .**

(2) If Δ is a union of a subdiagram Δ' and a rim θ -band \mathcal{T} of base width $\leq 2LN$, then there are at most $4LN$ a -edges on the boundaries of (θ, q) -cells of \mathcal{T} by (iii) (a), and so $|\mathbf{top}(\mathcal{T})| - |\mathbf{bot}(\mathcal{T})| \leq 4LN\delta$ by Lemma 5.20(a). Therefore $|\partial\Delta'|_a \leq 4LN\delta + |\partial\Delta|_a$ but $|\partial\Delta'|_\theta = |\partial\Delta|_\theta - 2$. Hence $|\partial\Delta'| \leq |\partial\Delta| - 1$ by Lemma 5.20 (b) since $5LN < \delta^{-1}$.



If \mathcal{T} has m q -cells, then $n \geq m$, and so, by Lemma 5.20 (d), the number of cells in \mathcal{T} is at most $\delta^{-1}n + 2m \leq 2\delta^{-1}n$ by Lemma 5.20 (d). Also we have $\mu(\Delta') \leq \mu(\Delta)$ and $\nu_J(\Delta') \leq \nu_J(\Delta)$ by Lemma 6.1 (b), and $n'_Q \leq n_Q$ by Lemma 5.6. Therefore by the inductive hypothesis for Δ' ,

$$\text{Area}(\Delta) \leq F(\Delta') + 2\delta^{-1}n \leq F(\Delta) - c_6(2n - 1) + 2\delta^{-1}n \leq F(\Delta)$$

since $c_6 > 2\delta^{-1}$; a contradiction. **Therefore Δ has no rim θ -bands of base width at most $2LN$.**

(3) Assume that Δ has a subcomb $\bar{\Delta}$ of base width $15N$. Hence we can apply Lemma 11.8 to the comb $\bar{\Delta}$ and consider two arising cases.

(a) $\bar{\Delta}$ admits a long quasicomb Γ such that

$$\text{Area}(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma) + c_3(\nu_J(\bar{\Delta}) - \nu_J^c(\bar{\Delta} \setminus \Gamma))$$

We multiply the right hand side by the number $c_4c_2^{-1} > 1$ and then replace the two coefficients $c_3c_4c_2^{-1}$ by bigger coefficients c_6 and c_5 , resp.; this is legal since $\nu_J(\bar{\Delta}) - \nu_J^c(\bar{\Delta} \setminus \Gamma) \geq 0$ by Lemma 7.3(e) and Remark 9.5, and $[\Gamma] \geq 0$ since Γ is a long subcomb. Hence

$$\text{Area}(\Gamma) \leq c_6[\Gamma] + c_4\mu^c(\Gamma) + c_5(\nu_J(\bar{\Delta}) - \nu_J^c(\bar{\Delta} \setminus \Gamma)) \quad (13.78)$$

Let $\mathbf{y} = \mathbf{y}^\Gamma$ and $\mathbf{z} = \mathbf{z}^\Gamma$. Since Γ is long, the compliment diagram $\Delta' = \Delta \setminus \Gamma$ satisfies $|\partial\Delta'| \leq |\partial\Delta| - |\mathbf{z}| + |\mathbf{y}| < |\partial\Delta|$. By Lemma 7.3 (a,e) and Remark 9.5, we also have $\mu(\Delta) \geq \mu(\Delta') + \mu^c(\Gamma)$ and $\nu_J(\bar{\Delta}) - \nu_J(\bar{\Delta} \setminus \Gamma) \leq \nu_J(\Delta) - \nu_J(\Delta')$. Since $\text{Area}(\Delta') \leq F(\Delta')$ by the inductive hypothesis, it follows from (13.78) that

$$\begin{aligned} \text{Area}(\Delta) &\leq \text{Area}(\Delta') + c_6[\Gamma] + c_4\mu^c(\Gamma) + c_5(\nu_J(\Delta) - \nu_J(\Delta')) \leq c_6(n - (|\mathbf{z}| - |\mathbf{y}|))^2 + \\ &\quad c_4\mu(\Delta') + c_5\nu(\Delta') + c_7n_Q f(T_{i-1}) + c_6[\Gamma] + c_4\mu^c(\Gamma) + c_5(\nu_J(\Delta) - \nu_J(\Delta')) \leq \\ &\quad c_6n^2 - c_6n(|\mathbf{z}| - |\mathbf{y}|) + c_6n(|\mathbf{z}| - |\mathbf{y}|) + c_4\mu(\Delta) + c_5\nu(\Delta) + c_7n_Q f(T_{i-1}) \leq F(\Delta) \end{aligned}$$

since $|\mathbf{y}| < |\mathbf{z}| \leq n$, $c_3 < c_6$, $c_3 < c_5$, and $c_2 < c_4$. Therefore Δ is not a counter-example, a contradiction.

(b) Δ has subcomb whose handle \mathcal{C} is a t - or t' -band with length l satisfying $T_j \leq l < 200T_j$ for some j , and \mathcal{C} separates a subcomb Γ of base width at most $14N$ from Δ . By Remark 7.1 applied to Γ , we have $n > l$. Now since $T_j \leq l < n \leq f(n) \leq T_i$, we have $j \leq i - 1$. Again let Δ' be the diagram $\Delta \setminus \Gamma$. Let $\mathbf{y} = \mathbf{y}^\Gamma$ and $\mathbf{z} = \mathbf{z}^\Gamma$. Then we have

$$|\partial\Delta'| \leq |\partial\Delta| - (|\mathbf{z}| - |\mathbf{y}|) \leq |\partial\Delta| - 2 \quad (13.79)$$

since the handle \mathcal{C} of Γ is passive, and so Γ is a long subcomb. Since Γ has a q -band \mathcal{C} , we immediately obtain

$$n_Q - n'_Q \geq 2 \quad (13.80)$$

for the numbers of q -edges in $\partial\Delta$ and in $\partial\Delta'$, and

$$\nu_J(\Delta) - \nu_J(\Delta') \geq 0 \quad (13.81)$$

by Lemmas 7.3(b) and 7.2(b).

By Remark 12.7 (2),

$$\mu(\Delta') \leq \mu(\Delta) + l^2/2 < \mu(\Delta) + (200T_{i-1})^2/2 \leq \mu(\Delta) + f(T_{i-1}) \quad (13.82)$$

since $(\delta')^{-1} > 2 \times 10^4$. Now, from the definition of the function F and Inequalities (13.79), (13.81), and (13.80), we get

$$\begin{aligned} F(\Delta) &\geq F(\Delta') + c_6(n^2 - (n - (|\mathbf{z}| - |\mathbf{y}|))^2) + c_5(\nu(\Delta) - \nu(\Delta')) + c_4(\mu(\Delta) - \mu(\Delta')) \\ &\quad + c_7(n_Q - n'_Q)f(T_{i-1}) \geq F(\Delta') + c_6n(|\mathbf{z}| - |\mathbf{y}|) + c_4(\mu(\Delta) - \mu(\Delta')) + 2c_7f(T_{i-1}) \end{aligned}$$

which together with Inequality (13.82) implies

$$F(\Delta) \geq F(\Delta') + c_6n(|\mathbf{z}| - |\mathbf{y}|) - c_4f(T_{i-1}) + 2c_7f(T_{i-1}) \geq \text{Area}(\Delta') + c_6[\Gamma] + c_7f(T_{i-1}) \quad (13.83)$$

because $\text{Area}(\Delta') \leq F(\Delta')$ by (13.79) and the minimality of the counter-example Δ .

On the other hand, by Lemmas 7.13, 5.20 (d), and inequality $|\mathbf{y}| = l < 200T_{i-1}$, we get

$$\begin{aligned} \text{Area}(\Gamma) &\leq 60N|\mathbf{y}|^2 + 2\delta^{-1}|\mathbf{z}||\mathbf{y}| = (60N + 2\delta^{-1})|\mathbf{y}|^2 + 2\delta^{-1}|\mathbf{y}|(|\mathbf{z}| - |\mathbf{y}|) \leq \\ &\quad (60N + 2\delta^{-1})|\mathbf{y}|^2 + 2\delta^{-1}[\Gamma] \leq 200^2 \times (60N + 2\delta^{-1})T_{i-1}^2 + 2\delta^{-1}[\Gamma] \leq c_6[\Gamma] + c_7f(T_{i-1}) \end{aligned} \quad (13.84)$$

because $c_6 > 2\delta^{-1}$, $c_7 \geq 10^6$, and $f(T_{i-1}) \geq \delta^{-1}T_{i-1}^2$. Now Inequalities (13.83, 13.84) yield

$$\text{Area}(\Delta) = \text{Area}(\Delta') + \text{Area}(\Gamma) \leq F(\Delta) - c_6[\Gamma] - c_7f(T_{i-1}) + \text{Area}(\Gamma) \leq F(\Delta),$$

a contradiction. **Hence Δ has no subcombs of base width $15N$.**

(4) Assume that Δ has a one-Step subcomb Γ whose handle is t or t' -band. By (3), we may assume that its base width is less than $15N$. Then we can use Lemma 9.2(2) and come to a contradiction as in (3)(a) above.

By (2)-(4), **the diagram Δ is solid. By Lemma 12.1(b), it has a hub.**

(5) Suppose we have a hub π and a crescent $\Psi = cl(\pi, \mathcal{C}_1, \dots, \mathcal{C}_l)$ given by Lemma 12.4. Assume that $|\mathbf{p}(\Psi)| > 2LN \max_{i=1}^l h_i$, where h_i is the length of \mathcal{C}_i . Then by Lemma 12.3, $|\partial\Psi'| < |\partial\Delta|$ for the subdiagram $\partial\Psi' = \Delta \setminus (\pi \cup \Psi)$. Besides it follows from the definition of crescent that $n'_Q < n_Q$, where n'_Q is the number of q -edges in $\partial\Psi'$. Since by the inductive hypothesis $\text{Area}(\Psi') \leq F(\Psi')$, we obtain by Lemma 12.3 that

$$\text{Area}(\Delta) \leq F(\Psi') + c_4(\mu(\Delta) - \mu(\Psi')) + c_5(\nu_J(\Delta) - \nu_J(\Psi')) + c_6n(n - |\partial\Psi'|) \leq F(\Delta),$$

and so Δ is not a counter-example.

(6) Now we assume that Δ has a crescent Ψ and a hub as in (5), but now $|\mathbf{p}(\Psi)| \leq 2LN \max_{i=1}^l h_i$. If the conditions of Lemma 12.8 are satisfied, then that lemma leads to a contradiction as in case (5) above since $|\partial\Delta'| < |\partial\Delta|$ by Lemma 12.6(a), $n'_Q < n_Q$, and the diagram $\Delta_1 \cup \Delta'$ (with notation of Section 12) has the same boundary label as Δ (see Lemma 5.2). Similarly we obtain a contradiction under assumptions of Lemma 12.9, if we cut off the subdiagram $\Delta_1(0)$ with the spokes $\mathcal{C}_2, \dots, \mathcal{C}_l$ since these spokes also bound a crescent $\Psi_{2,l}$ by Lemma 12.4.

(7) Thus it remains to assume that the maximal h_i for the crescent, say h_1 (since the case with $\Psi_{2,l}$ from Lemma 12.9 is absolutely similar) **satisfies inequalities $T_j \leq h_1 < 9T_j$ for some j and $|\mathbf{p}| = |\mathbf{p}(\Psi)| \leq 2LNh_1$** . Notice that $j \leq i-1$ because, by Lemma 13.1 and the assumption $f(n) \leq T_i$, we have $T_j \leq h_1 < \text{Area}(\Delta) \leq f(n) \leq T_i$.

Now we will use the notation of Lemma 12.6. By Lemma 12.2, we have

$$\begin{aligned} \text{Area}(\Delta_1) &\leq 2\text{Area}(\Psi) + 1 \leq 4(2LN(2h_1) + \delta^{-1}(2LNh_1))h_1 + 1 \leq \\ (16LN + 8\delta^{-1}LN)h_1^2 + 1 &< 9\delta^{-1}LNh_1^2 < 800\delta^{-1}LNT_{i-1}^2 \leq 800LNf(T_{i-1}) \end{aligned} \quad (13.85)$$

By Lemma 12.6 (c),

$$\mu(\Delta') \leq \mu(\Delta) + 2h_1^2 < \mu(\Delta) + 200T_{i-1}^2 \leq \mu(\Delta) + f(T_{i-1}),$$

and so by Lemma 12.6 (b) and the definition of $\mu(*)$,

$$\mu(\Delta) - \mu(\Delta') = c_0(\kappa(\Delta) - \kappa(\Delta')) + (\lambda(\Delta) - \lambda(\Delta')) \geq -f(T_{i-1}) \quad (13.86)$$

Now using Lemma 12.6 (a, d), inequalities $n_Q \geq n'_Q + 2$ and (13.86) we have

$$F(\Delta) - F(\Delta') > c_6 \times 0 - c_4f(T_{i-1}) + c_5 \times 0 + 2c_7f(T_{i-1}) \geq c_7f(T_{i-1})$$

This inequality, (13.85), the inductive hypothesis (valid by Lemma 12.6 (a)), and Lemma 5.2 imply

$$\text{Area}(\Delta) \leq \text{Area}(\Delta') + \text{Area}(\Delta_1) < F(\Delta') + 800LNf(T_{i-1}) \leq F(\Delta') + c_7f(T_{i-1}) < F(\Delta),$$

and so Δ is not a counter-example in this case too.

The proof is complete. □

Now we go back to the combinatorial length $\|\cdot\|$ and make use of the obvious quadratic upper bounds for the mixtures.

Lemma 13.3. *There is a constant C such that for every $i = 2, 3, \dots$ and arbitrary minimal diagram Δ with $\|\partial\Delta\| = r$, we have $\text{Area}(\Delta) \leq C(r^2 + rT_{i-1}g(CT_{i-1}^2)^2 + T_{i-1}^4)$ provided $Cr(g(Cr^2)^2 + Cr^3) < T_i$.*

Proof. By lemma 5.20, $|\partial\Delta| = n \leq r \leq \delta^{-1}n$. Recall also that by Lemma 6.1 (a) and the definition of μ - and ν_J -mixtures, $\mu(\Delta) \leq (c_0 + 1)r^2$, $\nu_J(\Delta) \leq Jr^2$, and also $n_Q \leq n$. Now the statement of the lemma follows with a constant $C \geq 2\delta^{-1}c_7$ from Lemma 13.2 and from the definitions of the function f . □

Finally we apply Theorem 14.1 converted into Property (xviii) of trapezia.

Lemma 13.4. *The Dehn functions of the groups G and M are almost quadratic.*

Proof. We consider only the Dehn function $d(r)$ of the group G since a simpler proof works for M . (One considers only diagrams having no hubs in the later case.)

Assume that an integer m satisfies the hypothesis of (xvii) and m is large enough, say $m > \max(T_1, C^{20}(\log m)^{40})$, where C is provided by Lemma 13.3. There is a maximal i such that $T_{i-1} \leq m$. Since T_{i-1} is the height of a standard trapezium with some bottom W and any rule corresponding to this trapezium can decrease the length of the input sector at most by 1, we have $|W|_a \leq T_{i-1} \leq m$, and so by (xvii) and the choice of i , we get

$$\exp T_{i-1} < m < T_i, \quad (13.87)$$

It follows from (13.87) and the choice of m that

$$CT_{i-1}^2 < m^{1/20}. \quad (13.88)$$

Now, on the one hand, inequality $n < m/7$ implies

$$g(n) \leq \frac{4}{7}m + 3 \log m < m < T_i \quad (13.89)$$

by (xvii). On the other hand, by Property (xix), any value $g(n)$ of the function g either belongs to some interval $(T_j, 9T_j)$ or $g(n) \leq 6n$. By (13.89), we have $j < i$ in the former case. Therefore if $n < m/7$, then in any case

$$g(n) \leq \max(9T_{i-1}, 6n) \leq \max(9 \log m, 6n) \quad (13.90)$$

Hence there is a constant D such that for every integer r such that $r > 9 \log m$ and $Dr^5 \leq m$, we have

$$Cr(g(Cr^2)^2 + Cr^3) < Dr^5 \leq m < T_i \quad (13.91)$$

Now by Inequalities (13.91), (13.87), (13.90), (13.88), and by Lemma 13.3, we have

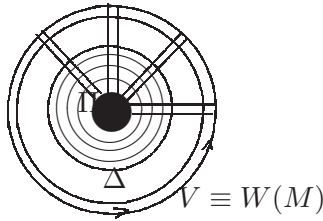
$$d(r) \leq C(r^2 + rT_{i-1}g(CT_{i-1}^2)^2 + T_{i-1}^4) \leq C(r^2 + r(\log m)g(CT_{i-1}^2)^2 + T_{i-1}^4) \leq$$

$$C(r^2 + r(\log m)(\max(9 \log m, 6m^{1/20}))^2 + (\log m)^4) \leq 2Cr^2$$

if $Dr^5 \leq m$, and $r > m^{1/6}$.

Since the set of integers m satisfying the hypothesis of (xvii), is infinite by (xviii), we can find for almost every such m , an integer r satisfying inequalities $Dr^5 \leq m < r^6$, and so the inequality $d(r) \leq 2Cr^2$ holds on an infinite set of integers r . \square

End of proofs of Theorems 1.1 and 1.3. Using the notation of Lemma 4.45, we consider a word $V \equiv W(M)$ for an arbitrary admissible input word W of the machine M_4 . Assume that $V = 1$ in the group G . Then there is a minimal diagram Δ whose boundary path is labeled by V . Since every state letter from the vector of start states of M occurs in V exactly once, every maximal q -band of Δ must end on a hub, and Δ has $m \geq 1$ hubs. On the other hand, $m \leq 1$ by Lemma 5.19, since $|V|_q = LN$ by the definition of the standard base for the machine M . Thus Δ has exactly one hub Π , and so every maximal q -band of Δ connects the boundaries of Π and Δ .



Since V has no θ -edges, by Lemma 5.6, every non-hub cell of Δ belongs to a θ -annulus surrounding the hub Π . (The set of these annuli is not empty since V has no state letters of the hub relation.) Hence one can remove Π , make a radial cut, and construct a trapezium with base (4.6). By Lemma 5.10 (1), the computation of M corresponding to this trapezium accepts the word V , and therefore $V \in X_5$ by Lemma 4.45.

Conversely, assume that $V \equiv W(M) \in X_5$. Then by Lemma 5.10 (2), there is a trapezium with base (4.6) corresponding to an accepting computation $W(M) \rightarrow \dots$ of M . Now one may identify the left-most and the right-most maximal t -bands of this trapezium and paste up the hole of the obtained annular diagram by a hub. Hence V is a boundary label of a disc van Kampen diagram, and therefore $V = 1$ in G .

The obtained criterion shows that the word problem is undecidable for G since the set X_5 is not recursive by Lemma 4.45. By Lemma 13.4, the proof of Theorem 1.1 is complete.

Relations (5.7) of the group M define the structure of a (multiple) HNN-extension on the group M whose base is the free subgroup generated by all a - and q -letters, and for every rule, one has a stable θ -letter. (See the presentation of every S-machine as an HNN-extension in [15].) The statements 2 and 3 of Theorem 1.3 hold for M by Lemma 13.4 and by Step 1 of the proof of Lemma 13.1. Finally, a word $V \equiv W(M)$ is conjugate to the hub in M iff $V \in X_5$. (The proof is similar to the criterion obtained above for the equality $W(M) = 1$ in G , but now one considers annular diagrams over M instead of disc diagrams over G .) Now the statement 1 of Theorem 1.3 follows from Lemma 4.45, and the proof is complete.

Theorem 13.5. *There exists a finitely presented group G with almost quadratic Dehn function $d(n)$ such that $d(n) \geq \exp n$ for infinitely many n -s, and $d(n)$ is bounded from above on the entire \mathbb{N} by an exponential function.*

Proof. We will make a few alternations in the proof of Theorem 1.1.

Given a word a^n , it is easy to check in linear time whether $n = 2^m$ for some natural m or not and to compute $m = \log_2 n$ if $m \in \mathbb{N}$. Therefore there is a deterministic Turing machine M_0 with linear time complexity which accept a word a^n iff n belongs to the sequence $n_1 = 1$, $n_i = 2^{2^{n_{i-1}}}$ for $i > 1$. Clearly almost every n_i is an h_α -good number for any function $h_\alpha(x) = 2^{2^{\alpha x}}$, and we can use this property instead of Theorem 14.1.

Starting with M_0 , we construct the machines M_1, \dots, M_4, M and define the group G as in the paper. Then we obtain, as in Theorem 1.1, that $d(n)$ is almost quadratic (since the non-recursiveness from Theorem 14.1 has never been used for this goal).

For some positive constants c' and c'' , Lemmas 4.16(b) and 4.25(b) give the estimates $\exp(c'n_i) < T_i < \exp(c''n_i)$ for the time of acceptance T_i of the word a^{n_i} by the machine M_3 . As in the above “End of proofs”, it follows that the length of the corresponding to a^{n_i} accepted input word V_i of the machine M is $O(n_i)$ while the area is at least $T_i > \exp(c'n_i)$. Thus $d(n)$ is bounded from below on the infinite sequence of n_i -s by an exponent. It remains to obtain an exponential upper bound $d(n) < \exp(Cn)$ on the entire \mathbb{N} . (We do not need any mixtures for this goal.)

Assume that C is large enough, and a minimal diagram Δ over G has area $\geq \exp(Cn)$. Then Δ has no rim θ -bands \mathcal{T} of base width $\leq 2LN$ because $|\partial\Delta'| < n$ for the subdiagram $\Delta' = \Delta \setminus \mathcal{T}$ in Case (2) of the proof of Lemma 13.2. Similarly Δ has no long subcombs (or subquasicombs) since Lemma 13.1 (Step 1 of the proof) provides us with a cubic upper bound of the area of any subcomb (as function of n .) Therefore the diagram Δ is solid, and therefore Lemma 12.6 (a) reduces our task to diagrams having at most one hub. Indeed, by Lemma 5.19, The number of hubs in Δ does not exceed n , and the functions $\exp(Cn)$ and $n \exp(Cn)$ are equivalent.

We may assume that Δ has exactly one hub since otherwise its area is bounded by a cubic function of the perimeter. Now by Lemma 12.2 applied to the whole Δ , we

conclude that every maximal θ -band of Δ is an annulus, and so, as at the “End of proofs” above, the boundary label of Δ is of the form $V \equiv W(M)$ for some admissible input word of the machine M_4 . Therefore it suffices (by Lemma 5.10) to find an exponential upper bound for the accepting computations of M_4 with respect to the length $\|W\|$ of an input admissible word W . Such an upper bound (even a linear bound) is given by Lemma 4.38 if the length of the reduced computation of W does not belong to any interval $(T_i, 9T_i)$. The argument of that lemma works in other cases if the computation does not contain the standard computation of length T_i , (i.e., the computation of n_i in our situation). However the proof of Lemma 4.38 also shows that $\|W\| \geq n_i$ in the remaining cases. Therefore the length of the computation has the exponential upper bound $9T_i \leq 9 \exp(c''\|W\|)$, and the proof is complete since $C \gg c''$.

Remark 13.6. One can replace the exponential function by a multiexponential one or by many other functions with at least exponential growth in the formulation and in the proof of Theorem 13.5.

□

14 Appendix: A very sparse immune set

By M.V.Sapir

Let X be a recursively enumerable (r.e.) language in the binary alphabet recognized by a Turing machine M . If $x \in X$ then the *time* of x (denoted $\text{time}(x)$ or $\text{time}_M(x)$) is, by definition, the minimal time of an accepting computation of M with input x . For any increasing function $h: \mathbb{N} \rightarrow \mathbb{N}$, a real number m is called *h -good* for M if for every $w \in X$, $\|w\| < m$ implies $h(\text{time}(w)) < m$.

For every number n , the number of digits in n is denoted by $\|n\|_2$. This number is roughly $\log_2 n$. Since we are not using any other logarithms in Appendix, we shall omit 2 in \log_2 . Similarly, we shall write $\exp x$ for 2^x .

The proof of the following theorem uses an idea communicated to the author by S.Yu. Podzorov. For every $\alpha > 0$, let $h_\alpha(n) = \exp(\alpha n)$.

Theorem 14.1. *There exists a Turing machine M_0 recognizing a r.e. non-recursive set X such that the set of all h_α -good numbers for M_0 is infinite for all $\alpha > 0$.*

Proof. We use a recursive enumeration of all Turing machines from [8]. By Matiyasevich's solution of the 10th Hilbert problem [10], there exists a polynomial $F(a, b, x_1, \dots, x_s)$ with integer coefficients such that a is recognized by the Turing machine number b if and only if $F(a, b, x_1, \dots, x_s) = 0$ for some natural numbers x_1, \dots, x_s . We are going to use Gödel numeration of s -tuples of natural numbers. For every natural m let $\mathbf{g}(m)$ be the s -tuple having Gödel number m . Note that all coordinates of this tuple do not exceed m and the time to compute $\mathbf{g}(m)$ is linear in $\|m\|_2$.

Note also that if $\|a\|_2, \|b\|_2, \|x_i\|_2 \leq n$ ($i = 1, \dots, s$) then the time needed to compute $F(a, b, x_1, \dots, x_s)$ is bounded by a polynomial in n depending only on F . Also the time to compute binary value of the exponent $\exp n$ (given n written in binary) is linear in n (and exponential in $\|n\|_2$).

The algorithm of enumerating elements in X involves auxiliary formulas for functions $f_m: \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$, and two sequences of numbers $b(m), x(m)$, $m = 0, 1, 2, \dots$.

Step 0. Set $f_0(i) = i$ (that is by definition f_0 is the identity function), $b(0) = 0$, $x(0) = 0$, $X = \emptyset$.

Step $m \geq 1$. Search for the minimal $i = i(m) \leq m$ such that for some $m' \leq m$ $F(f_{m-1}(i), i, \mathbf{g}(m')) = 0$ and $f_{m-1}(i)$ is not already in X . If such an i exists, add $f_{m-1}(i)$ to X , compute the new numbers $b(m) = \exp \exp \exp \exp(m + \|b(m-1)\|_2 + x(m-1))$ (four exponents), $x(m) = \max(x(m-1), f_{m-1}(i))$, and define the function f_m by adding in the definition of f_m that $f_m(j) = j + b(m)$ for every $j > i$. (Note that $f_m(j) = f_{m-1}(j)$ for every $j \leq i$.) In that case we say that the step m was *successful*, and i is *responsible* for counting $f_{m-1}(i)$ into X . Otherwise (if either $i(m)$ does not exist or $f_{m-1}(i)$ is already in X) let $f_m = f_{m-1}$, $b(m) = b(m-1)$, $x(m) = x(m-1)$. Then go to the next step. Note that for every i, m we have $f_m(i) \leq i + b(m)$. Therefore if step m is successful, we have $x(m) \leq x(m-1) + m + b(m-1)$. Hence

$$x(m) < b(m). \quad (14.92)$$

Inequality (14.92) holds also for unsuccessful m if m is larger than the number of the first successful step by induction, because in that case $x(m) = x(m-1)$, $b(m) = b(m-1)$.

We claim that every number $i \geq 1$ is responsible for at most 2^i members of X . Indeed, every i can be responsible only for numbers of the form $f_m(i)$. The value $f_m(i)$ can differ from $f_{m-1}(i)$ only if some number $i' < i$ is responsible for counting some number $f_{m'-1}(i')$ into X at some step $m' > m \geq i$. Therefore we have $f_1(1) = f_2(1) = \dots$, so 1 can be responsible only for at most one number in X . This implies that 2 can be responsible for at most two numbers only: the value $f_0(2)$ is 2, and the value $f_m(2)$ can differ from $f_{m-1}(2)$ only if 1 is responsible for some number in X . Similarly, the value $f_m(i+1)$ can differ from $f_{m-1}(i+1)$ only when a number $j \leq i$ becomes responsible for counting a number into X . By induction it can happen at most $1 + 2 + 2^2 + \dots + 2^i = 2^{i+1} - 1$ times. Therefore $i + 1$ can be responsible for at most 2^{i+1} numbers in X as claimed.

Let us prove now that the set X is what we need. It is clear that X is recursively enumerable: the machine M enumerating this set is described in the definition of X . (Recall, that a Turing machine enumerating a set of words X in a finite alphabet differs from a Turing machine recognizing it: it does not have input sector and the accept configuration. It starts working with all tapes empty, and writes words from X in the first tape one by one, separated by a special symbol. After a new word is written in tape 1 (i.e. when the machine *counts a new word into* X), the machine puts the separating symbol next to that word and continues working. If X is infinite, the machine works infinitely long. For every $u \in X$, we can talk about the *time to count it into* X , i.e. the shortest length of the computation after which the word first appears in the first tape.)

Let us prove that X is not recursive. Suppose the contrary - that X is recursive. Then its complement is recursively enumerable. Therefore there exists a natural number b such that

$$(*) \ F(a, b, x_1, \dots, x_s) = 0 \text{ for some } x_1, \dots, x_s \text{ if and only if } a \text{ is } \mathbf{not} \text{ in } X.$$

Let b be the number from (*). There exists $m \geq 1$ which is bigger than the number of the first successful step, and such that for every $m' \geq m$ either $i(m') > b$ or the step number m' is not successful (this follows from the fact that each i is responsible for finitely many members of X only). Let m be one of the numbers with this property.

By definition,

$$f_{m'}(j) = f_m(j) \quad (14.93)$$

for every $m' > m, j < b$.

Claim. No $j \neq b$ can be responsible for counting $r = f_{m-1}(b)$ into X .

Suppose that $j < b$ is responsible for counting r into X . That cannot happen at step $m' > m$ because $i(m') > b$ since $r = f_{m-1}(j) \leq f_{m'-1}(j) < f_{m'-1}(i(m'))$ by definition of m . If that happens at step m , then $f_{m-1}(j) = r = f_{m-1}(b)$ which is impossible since f_{m-1} is strictly increasing. If that happens at step number $m' < m$, then, since $j < b$, $f_{m'}(b) \geq b + b(m') \geq b(m')$, and we would have

$$f_{m-1}(b) \geq f_{m'}(b) \geq b(m') > x(m') \geq r$$

by (14.92) and because $x(m')$ is the maximum of all numbers counted into X at steps $\leq m'$, including r , a contradiction.

Suppose that $j > b$ is responsible for counting r into X at some step m' . Suppose that $m' < m$. We have $f_{m'-1}(j) = r$. Since $j \neq b$, $f_{m'-1}(b) \neq r$. Therefore $f_k(b)$ has changed at some step k such that $m' \leq k \leq m-1$. Hence there exists a successful step number k , $m' \leq k \leq m-1$ such that $i(k) < b$. But then

$$f_{m-1}(b) \geq f_k(b) \geq b(k) > x(k) \geq x(m') \geq r,$$

a contradiction.

It remains to consider the case when $j > b, m' \geq m$. But in that case (since $f_{m'-1}$ is strictly increasing)

$$f_{m'-1}(j) > f_{m'-1}(b) \geq f_{m-1}(b) = r,$$

a contradiction. This completes the proof of our claim.

Now if r is in X then for some m' , $F(r, b, \mathbf{g}(m')) = 0$ (since by the Claim only b can be responsible for counting r into X). But this would mean, by the choice of b (see (*)), that r is **not** in X , a contradiction. On the other hand if r is not in X then $F(r, b, \mathbf{g}(j)) = 0$ for some j , therefore at some step m' , b would be responsible for counting r into X , so $r \in X$, a contradiction. This shows that X is not recursive. In particular, X is infinite.

Note that there exists a deterministic Turing machine M_0 which recognizes X and such that for every $m \in X$, the time to recognize it by M_0 is linearly bounded in terms of the time to count it into X by M . Indeed let us add the input tape to the tapes of M . The machine M_0 will execute M on its tapes. Every time there is a new word counted into X , the machine M_0 checks whether this word coincides with the input word. After the match is found, M_0 erases all tapes and stops.

Now let us determine the h_α -good numbers of the machine M_0 . We say that a number n is *appropriate* if $i(n)$ exists, $f_{n-1}(i(n))$ is counted into X at step n and none of $i(n')$ with $n' > n$ is smaller than $i(n)$. Clearly the set of appropriate numbers is infinite (since every number is responsible only for finitely many members of X , see above). Let B be the set of numbers $\lfloor \log b(n) \rfloor$ for appropriate n . Let us show that almost all numbers in B are h_α -good.

Indeed, let us estimate the time of a number $r = f_{n-1}(i_n)$ counted into X at an appropriate step number n by M_0 . The total number of evaluations of F needed for this is at most n^3 ($\leq n$ steps, at most $n \times n$ evaluations of $F(f_{n'-1}(i), i, \mathbf{g}(t))$ at each step where $1 \leq i \leq n, 1 \leq t \leq n$). We can estimate the time of each evaluation of F as a

polynomial in $\|n\|_2 + \|b(n-1)\|_2$. In addition of computing values of F , we also have to compute the numbers $b(n')$ and the formulas for $f_{n'}$ (at most n times). The time of computing $b(n')$ and $f_{n'}$ does not exceed the time of computing $b(n-1)$ and f_{n-1} . And those times can be bounded by a polynomial in $\|b(n-1)\|_2$. Recall that the time of recognizing r by M_0 is bounded by a constant times the time of counting r into X by M . Thus the total time of accepting r by the machine M_0 is bounded by $cn^c\|b(n-1)\|_2^c + c$ for some constant c . Note that

$$\exp \exp(cn^c\|b(n-1)\|_2^c + c) < \exp \exp \exp(\|b(n-1)\|_2) < \log b(n) \quad (14.94)$$

for almost all n . Also notice that since n is appropriate, by the definition of f_n , there are no numbers $r' \in X$ between $r+1$ and $b(n)$. Hence if $r \in X$ and $\|r\|_2 < \log b(n)$ then $\exp \exp(\alpha \text{time}(r)) < \log b(n)$ by (14.94) (for all but finitely many n and some α). Hence $\log b(n)$ is an h_α -good number of M_0 for almost all n . Therefore the set of h_α -good numbers for M_0 is infinite. \square

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